Interpretation of Differentials

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We frequently solve geometrical and physical problems by obtaining an approximate expression for differential dP in terms of differential dQ and then integrating dP to obtain P. We assume that the expression for P is exact even though we used an approximate formula for dP. This is justified by saying that the differentials are infinitely small quantities. For example, when we derive an expression for the area of a circular disc (see example 1) we set $dA = 2\pi r dr$ which is an approximate expression when the diffentials are interpreted as real numbers. In this article we try to define a method for computing P so that we don't need approximate expressions in the derivation.

Theorem 1. *Let $a, b \in \mathbb{R}$ and $a < b$. Suppose that*

$$\Delta f = g(x)\Delta x + h(x, \Delta x)$$

for all $x \in \mathbb{R}$ and $\Delta x \in \mathbb{R}_+$ for which $x, x + \Delta x \in [a, b]$. Suppose also that g is Riemann integrable and for all $\varepsilon \in \mathbb{R}_+$ there exists $R \in \mathbb{R}_+$ so that

$$\left| \frac{h(x, \Delta x)}{\Delta x} \right| < \varepsilon \tag{1}$$

for all $x, x + \Delta x \in [a, b]$, $0 < |\Delta x| < R$. Let $t \in [a, b]$. Let $n \in \mathbb{Z}_+$, and define $\Delta' x := \frac{t-a}{n}$ and $x_i := a + \frac{i}{n}\Delta' x$ $i \in \mathbb{N}$, $i \leq n$. Define also $\Delta f_i := f(x_{i+1}) - f(x_i)$ where $i \in \mathbb{N}$, $i < n$. Define

$$f(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta f_i$$

Now

$$f(t) = \int_a^t g(x)dx$$

and $df = g(x)dx$.

Proof. Let

$$f(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta f_i = \lim_{n\to\infty} \sum_{i=0}^{n-1} g(x_i)\Delta' x + \lim_{n\to\infty} \sum_{i=0}^{n-1} h(x_i, \Delta' x).$$

Now

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} g(x_i)\Delta' x = \int_a^t g(x)dx$$

by the definition of the Riemann integral.

Let $\varepsilon \in \mathbb{R}_+$. Choose $R_1 \in \mathbb{R}_+$ so that

$$\left|\frac{h(x, \Delta x)}{\Delta x}\right| < \frac{\varepsilon}{t-a}$$

for all $x, x+\Delta x \in [a,b]$, $0 < |\Delta x| < R_1$. Let $n \in \mathbb{Z}_+$ so that

$$n > \frac{t-a}{R_1}.$$

Now

$$\left|\sum_{i=0}^{n-1} h(x_i, \Delta' x)\right| < \sum_{i=0}^{n-1} \frac{\varepsilon}{t-a}|\Delta' x| = n\frac{\varepsilon}{t-a}\frac{t-a}{n} = \varepsilon.$$

Hence

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} h(x_i, \Delta' x) = 0.$$

□

Theorem 2. *A sufficient condition for inequality* (1) *is that there exist* $S, C \in \mathbb{R}_+$ *so that*

$$|h(x, \Delta x)| < C|\Delta x|^2$$

for all $x, x+\Delta x \in [a,b]$ *and* $0 < |\Delta x| < S$.

Proof. Set $R := \min\{S, \varepsilon/C\}$. Now

$$\left|\frac{h(x, \Delta x)}{\Delta x}\right| < C|\Delta x| < \varepsilon.$$

□

Note that we can't prove Theorem 1 directly by the Fundamental Theorem of Calculus because we would need to *assume* that there exists a function $f : [a,b] \to \mathbb{R}$ for which $f(x+\Delta x) - f(x) = \Delta f(x, \Delta x)$ in order to use it.

Example 1. *Derive an expression for the area of a disc whose inner radius is* r_a *and outer radius* r_b.

Solution: Define ΔA to be the area of a disc with inner radius r and width Δr. We have

$$2\pi r \Delta r \leq \Delta A \leq 2\pi(r + \Delta r)\Delta r$$

By setting $g(r) := 2\pi r$ and $h(r, \Delta r) := 2\pi(\Delta r)^2$ we get $A = \pi r_b^2 - \pi r_a^2$ by Theorems 1 and 2.

Example 2. *Suppose that a particle is moving under influence of a constant force* $F = ma$ *for time* T *and the particle is initially at rest. Derive an expression for the kinetic energy of the particle. Assume that the work done by a constant force* F *is* $W = Fs$ *where* s *is the distance that the particle moves in the direction of the force. Assume also that the kinetic energy of a particle at rest is* 0.

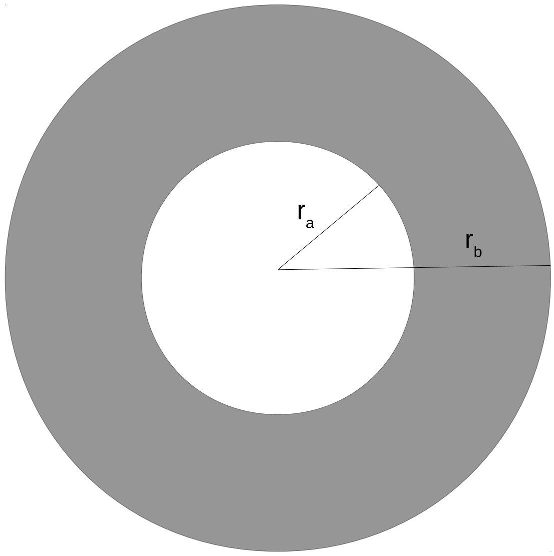

Figure 1: The area determined in example 1.

Solution: We define Δs to be the distance that the particle moves in the time interval $[t, t+\Delta t]$. We have $v = at$,

$$at\Delta t \leq \Delta s \leq a(t+\Delta t)\Delta t,$$

and

$$a(t+\Delta t)\Delta t = at\Delta t + a(\Delta t)^2.$$

Set $g(t) := at$ and $h(t, \Delta t) := a(\Delta t)^2$ and it follows from Theorems 1 and 2 that the distance that the particle moves in time T is

$$s = \int_0^T atdt = \frac{1}{2}aT^2$$

By setting $v_f = aT$ we obtain

$$E_k = W = \frac{1}{2}FaT^2 = \frac{1}{2}ma^2T^2 = \frac{1}{2}mv_f^2.$$