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Multiresolution analysis for compactly supported interpolating tensor product wavelets

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We construct multidimensional interpolating tensor product multiresolution analyses (MRA's) of the function spaces $C_0(\mathbb{R}^n, K)$, $K = \mathbb{R}$ or $K = \mathbb{C}$, consisting of real or complex valued functions on $\mathbb{R}^n$ vanishing at infinity and the function spaces $C_{\mathrm{u}}(\mathbb{R}^n, K)$ consisting of bounded and uniformly continuous functions on $\mathbb{R}^n$. We also construct an interpolating dual MRA for both of these spaces. The theory of the tensor products of Banach spaces is used. We generalize the Besov space norm equivalence from the one-dimensional case to our n-dimensional construction.

Keywords: Interpolating wavelets; multivariate wavelets; multiresolution analysis; tensor product; injective tensor norm; projective tensor norm; Besov space.

AMS Subject Classification: 46A32, 46B28, 15A69, 46E10

1. Introduction

Chui and Li[6] have constructed a one-dimensional MRA of function space $C_{\mathrm{u}}(\mathbb{R})$ (bounded and uniformly continuous complex valued functions on $\mathbb{R}$) for interpolating wavelets. Donoho[17] has derived convergence results for interpolating wavelets on space $C_0(\mathbb{R})$. Goedecker's book[20] contains also some material about multidimensional interpolating wavelets. Goedecker[20] uses the term interpolating wavelets to mean Deslauriers–Dubuc wavelets whereas some other authors such as Chui and Li[6] use the term to mean roughly wavelet families whose mother scaling function has the cardinal interpolation property $\varphi(k) = \delta_{k,0}$ for all $k \in \mathbb{Z}$ (one-dimensional case). We follow the latter convention in this paper. Chui and Li[5] have also constructed a general framework for multivariate wavelets. However, the theory in that paper uses function space $L^2(\mathbb{R}^n)$ and so it is unsuitable for the approach in this paper. Dubuc and Deslauriers[12,18] have investigated interpolation processes related to the Deslauriers–Dubuc fundamental functions. Han and Jia[23] have discussed fundamental functions (see the paper for a definition) satisfying the cardinal interpolation property. Numerical values for the wavelet filters of the Deslauriers–Dubuc wavelets have been given in Refs. 20 and 30.

Theory for orthonormal wavelets has been developed e.g., by Daubechies,[9,8] Meyer,[38] and Wojtaszczyk.[45] Kovačević and Sweldens[30] have investigated the use of digital filters for multidimensional biorthogonal wavelets. Lewis[36] has constructed a MRA of function space $L^2(\mathbb{R})$ for interpolating wavelets in one dimension. Lawton *et al.*[35] have presented conditions for the refinement mask of an orthonormal MRA. Ji and Shen[27] have given a condition for the refinement mask to be interpolatory and a condition for a dual refinement mask. Krivoshein and Skopina[31] have presented convergence results for frame-like wavelets that are not frames. Karakaz'yan *et al.*[29] have described symmetric interpolatory masks generating dual compactly supported wavelet systems and they have also given formulas for dual refinement masks. Shui *et al.*[43] have shown how to construct M-band wavelets with all the following properties: compact support, orthogonality, linear-phase, regularity and interpolation. This kind of scaling functions exist only when $M \geq 4$. Dahlke *et al.*[7] have developed a new method to construct higher-dimensional scaling functions. Their method is based on solutions to specific Lagrange interpolation problems by polynomials. Han[22] has investigated symmetries of refinable functions. DeVore *et al.*[13] have developed some theory for multidimensional orthonormal MRA of space $L^p(\mathbb{R}^d)$, $0 < p < \infty$, $d \in \mathbb{Z}_+$. He and Lai[25] have constructed nonseparable box spline wavelets on Sobolev spaces $H^s(\mathbb{R}^2)$. Wavelets have also been discussed in Ref. 24.

The theory about tensor products of Banach spaces has been presented e.g., in Ref. 41. We use the notation from Ref. 41 for Banach space tensor products in this paper. Schaefer's book[42] contains some material about tensor products of topological vector spaces. Reference 37 contains also an introduction to tensor products of Banach spaces. Domański *et al.*[15,16] have done some research on them. Michor[39] has represented tensor products of Banach spaces using category theory. Grothendieck[21] gives a classical presentation for tensor products of locally convex spaces. Theory for tensor products of Banach spaces can also be found in book.[11] Kustermans and Vaes[32] have developed some theory for space $C_0(G)$ where G is a compact or a locally compact group and for the tensor products of $C_0(G)$. Daws[10] has presented some material on tensor products of Banach algebras. Glöckner[19] has shown that the tensor products of topological vector spaces are not associative. Reinov[40] has investigated Banach spaces without the approximation property. Brodzki and Niblo[4] have done some research on the rapid decay property of discrete groups and the metric approximation property.

An introduction to Besov spaces can be found e.g., in Ref. 3. An introduction to Besov spaces using the Fourier transform based definition of these spaces can be found in Ref. 44. DeVore and Popov[14] have investigated the connection of Besov spaces with the dyadic spline approximation and interpolation of Besov spaces. Kyriazis and Petrushev[33] have given a method for the construction of unconditional bases for Triebel–Lizorkin and Besov spaces. The relationship between orthonormal wavelets and Besov spaces has been discussed by Meyer.[38] Almeida[1] has investigated wavelet bases in (generalized) Besov spaces.

Section 2 introduces some definitions used in the rest of this paper. We give some general definitions needed by the MRA's in Sec. 3. A multivariate MRA of $C_{\mathrm{u}}(\mathbb{R}^n, K)$, $K = \mathbb{R}$ or $K = \mathbb{C}$, is constructed in Sec. 4. A multivariate MRA of $C_0(\mathbb{R}^n, K)$, $K = \mathbb{R}$ or $K = \mathbb{C}$, is constructed in Sec. 5. The interpolating dual MRA is presented in Sec. 6. The Besov space norm equivalence from Donoho[17] is generalized for the n-dimensional interpolating MRA's in Sec. 7. A longer version of this paper with more detailed proofs can be found at Ref. 26.

2. Preliminaries

2.1. *General*

The set of all positive real numbers is denoted by $\mathbb{R}_+$ and the extended real line by $\mathbb{R}_*$. The set of all positive integer numbers is denoted by $\mathbb{Z}_+$. We define $\mathbb{R}_0 := \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_{0*} := \{x \in \mathbb{R}_* : x \geq 0\}$. When A and B are arbitrary sets, f is a function from A into B, and $X \subset A$ the image of X under f is denoted by $f[X]$. The set-theoretic support of a function $f : X \to \mathbb{C}$ where X is a set is denoted by $\mathrm{supp}_{\mathrm{set}}\, f$. The topological support of a function $f : T \to \mathbb{C}$ where T is a topological space is denoted by $\mathrm{supp}\, f$. When E is a metric space, $x \in E$, and $r \in \mathbb{R}_+$ the closed ball of radius r centred at x is denoted by $\overline{B}_E(x; r)$.

Definition 2.1. When $n \in \mathbb{Z}_+$ define

$$Z(n) := \{k \in \mathbb{Z}_+ : k \leq n\}.$$

We define the differences $\Delta_{\mathbf{h}}^m$ and $\Delta_{\mathbf{h}}$ as in Ref. 44 and Definition V.4.1 in Ref. 2.

Definition 2.2. Define

$$\left(\Delta_{\mathbf{h}}^1 f\right)(\mathbf{x}) := (\Delta_{\mathbf{h}} f)(\mathbf{x}) := f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{h} \in \mathbb{R}^n$, and $f \in \mathbb{C}^{\mathbb{R}^n}$ and

$$\Delta_{\mathbf{h}}^m f := \Delta_{\mathbf{h}}(\Delta_{\mathbf{h}}^{m-1} f)$$

for all $m \in \mathbb{N} + 2$, $\mathbf{h} \in \mathbb{R}^n$, and $f \in \mathbb{C}^{\mathbb{R}^n}$.

Definition 2.3. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let f be a function from $\mathbb{R}^n$ into K. Define

$$N_{\mathrm{cover}}(f) := \max_{\mathbf{x} \in \mathbb{R}^n} \#\{\mathbf{k} \in \mathbb{Z}^n : f(\mathbf{x} - \mathbf{k}) \neq 0\}.$$

Definition 2.4. Let E and F be normed vector spaces. When f is a compactly supported function from E into F define

$$r_{\mathrm{supp}}(f) := \inf\{r \in \mathbb{R}_0 : \mathrm{supp}_{\mathrm{set}}\, f \subset \overline{B}_E(0; r)\}.$$

Definition 2.5. Let $n \in \mathbb{Z}_+$. Define

$$I_{\text{trans}}(f, \mathbf{x}) := \{\mathbf{k} \in \mathbb{Z}^n : f(\mathbf{x} - \mathbf{k}) \neq 0\}$$

for all $f \in \mathbb{C}^{\mathbb{R}^n}$ and $\mathbf{x} \in \mathbb{R}^n$.

2.2. *Sequences and Cartesian products*

Definition 2.6. When $n \in \mathbb{Z}_+$ define

$$\mathbf{0}_n := (0)_{k \in Z(n)}.$$

Definition 2.7. When $n \in \mathbb{Z}_+$ define $J_+(n) := \{0,1\}^n \setminus \{\mathbf{0}_n\}$.

Definition 2.8. Define $\check{\mathbf{e}}_k := (\delta_{j,k})_{j \in \mathbb{Z}}$ for all $k \in \mathbb{Z}$. When $n \in \mathbb{Z}_+$ define $\mathbf{e}_k^{[n]} := \mathbf{e}_k^{Z(n)}$ for all $k \in Z(n)$.

2.3. *Vector spaces*

Suppose that A and B are topological vector spaces. We define $\mathcal{L}(A, B)$ to be the set of all continuous linear functions from A into B.

The term *operator* shall mean a bounded linear function from a normed vector space into another one. The term *projection* shall mean a linear projection of a vector space onto a vector space. The topological dual of a Banach space A is denoted by A^*. Unless otherwise stated A^* is equipped with the norm topology.

When A and B are Banach spaces we use the notation $A \subset_1 B$ to mean that A is isometrically embedded in B, i.e. $A \subset B$ and the inclusion map is distance preserving. When A and B are topological vector spaces we define $A =_{\text{tvs}} B$ to be true if and only if A and B are equal topological vector spaces. When B is a Banach space we use the notation $A \subset_{\text{c.s.}} B$ to mean that A is a closed subspace of B. When E is a Banach space, $A \subset E$, and $B \subset E^*$ we define $B \perp A$ to be true if and only if $\langle \tilde{b}, a \rangle = 0$ for all $a \in A$ and $\tilde{b} \in B$.

When E is a normed vector space and $x \in E$ we may use the notation $\|x|E\|$ to mean the norm of x in E. When B is a normed vector space and we write $A :=_{\text{n.s.}} \{x \in B : P(x)\}$ we assume that $\|x|A\| := \|x|B\|$ for each $x \in A$.

When E is a Banach space and A and B open or closed subspaces of E the topological direct sum of A and B is denoted by $A \dotplus B$.

Definition 2.9. Let E be a set and A be a vector space so that $A \subset E$. Suppose that $\|\cdot\|_A : E \to \mathbb{R}_{0*}$ is a norm on A. We say that the norm $\|\cdot\|_A$ *characterizes* A *on* E if and only if $\|x\|_A < \infty$ for all $x \in A$ and $\|y\|_A = \infty$ for all $y \in E \setminus A$.

2.4. *Function spaces and sequence spaces*

When $p \in [1, \infty]$, B is a Banach space, and I a denumerable set we denote the l^p space consisting of a subset of B^I by $l^p(I, B)$. As usual, $l^p(I) :=_{\text{n.s.}} l^p(I, \mathbb{C})$. When

$K = \mathbb{R}$ or $K = \mathbb{C}$, $n \in \mathbb{Z}_+$, and $p \in [1, \infty]$ we denote the L^p space consisting of a subset of the Borel functions from $\mathbb{R}^n$ into K by $L^p(\mathbb{R}^n, K)$. As usual, $L^p(\mathbb{R}^n) :=_{\text{n.s.}} L^p(\mathbb{R}^n, \mathbb{C})$. When X and Y are Borel spaces define $\operatorname{bor}(X, Y)$ to be the set of all Borel functions from X into Y.

Let $K = \mathbb{R}$ or $K = \mathbb{C}$. When T is a topological space we denote the space of K-valued bounded and continuous functions on T by $C_b(T, K)$. When E is a metric space the space of K-valued bounded and uniformly continuous functions on E is denoted by $C_{\text{u}}(E, K)$. When T is a locally compact Hausdorff space the space of K-valued bounded and continuous functions vanishing at infinity[34,17] is denoted by $C_0(T, K)$. The space of K-valued compactly supported and continuous functions on T is denoted by $C_{\text{com}}(T, K)$.

Definition 2.10. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and I be a denumerable set. Define

$$c_0(I, K) := \left\{ (a_\lambda)_{\lambda \in I} : \left(a_{\sigma(k)}\right)_{k=0}^{\infty} \in c_0(\mathbb{N}, K) \text{ for some bijection } \sigma : \mathbb{N} \twoheadrightarrow I \right\}$$

$$\|\mathbf{a}|c_0(I, K)\| := \|\mathbf{a}\|_\infty, \quad \mathbf{a} \in c_0(I, K).$$

Banach space $c_0(I, K)$ is isometrically isomorphic to $c_0(\mathbb{N}, K)$.

Definition 2.11. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let E be a closed subspace of $C_b(\mathbb{R}^n, K)$ for which the following condition is true:

$$\forall f \in C_b(\mathbb{R}^n, K), c \in \mathbb{R}_+, \mathbf{d} \in \mathbb{R}^n : f \in E \Leftrightarrow f(c \cdot -\mathbf{d}) \in E.$$

Let $a \in \mathbb{R}_+$, $\mathbf{b} \in \mathbb{R}^n$, and $\tilde{f} \in E^*$. The *a-dilatation and $\mathbf{b}$-translation of $\tilde{f}$*, denoted by $\tilde{f}(a \cdot -\mathbf{b})$, is defined by[6]

$$\langle \tilde{f}(a \cdot -\mathbf{b}), f \rangle := \frac{1}{a^n} \left\langle \tilde{f}, f\left(\frac{\cdot + \mathbf{b}}{a}\right) \right\rangle$$

for all $f \in E$.

Definition 2.12. Let $f \in \mathbb{C}^{\mathbb{R}^n}$, $r_1 \in \mathbb{R}_+$, and $\tilde{f} \in C\left(\overline{B}_{\mathbb{R}^n}(0; r_1)\right)^*$. Let $m \in \mathbb{N}$. We say that pair $(\tilde{f}, f)$ spans all the polynomials of degree at most m if and only if

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{f}, p(\cdot + \mathbf{k})|_{\overline{B}_{\mathbb{R}^n}(0; r_1)} \rangle f(\mathbf{x} - \mathbf{k}) = p(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and for all polynomials p of n variables that are of degree at most m.

We define the modulus of continuity in the standard way (see Definition V.4.2 in Ref. 2):

Definition 2.13. When $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}_+$, and $p \in [1, \infty]$ define

$$\omega_p^m(f; t) := \sup\{\|\Delta_{\mathbf{h}}^m f | L^p(\mathbb{R}^n)\| : \mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; t)\}$$

and

$$\omega(f; t) := \omega_\infty^1(f; t)$$

for all $f \in \operatorname{bor}(\mathbb{R}^n, \mathbb{C})$ and $t \in \mathbb{R}_0$.

The Besov spaces on $\mathbb{R}^n$ are denoted by $B^s_{p,q}(\mathbb{R}^n)$. We have $\mathcal{Z}^s(\mathbb{R}^n) =_{\mathrm{tvs}} B^s_{\infty,\infty}(\mathbb{R}^n)$ for all $s \in \mathbb{R}_+$ where $\mathcal{Z}^s(\mathbb{R}^n)$ are the Hölder–Zygmund spaces.[44] We also have $C^s(\mathbb{R}^n) =_{\mathrm{tvs}} \mathcal{Z}^s(\mathbb{R}^n) =_{\mathrm{tvs}} B^s_{\infty,\infty}(\mathbb{R}^n)$ when $s \in \mathbb{R}_+ \backslash \mathbb{Z}_+$ where $C^s(\mathbb{R}^n)$, $s \in \mathbb{R}_+ \backslash \mathbb{Z}_+$, are the Hölder spaces on $\mathbb{R}^n$.[44] When $m \in \mathbb{N}$ we denote the Banach space of functions with bounded and uniformly continuous partial derivatives up to mth degree equipped with the usual norm by $C^m(\mathbb{R}^n)$, see Ref. 44.

Definition 2.14. Let $n \in \mathbb{R}_+$, $s \in \mathbb{R}_+$, $p \in [1,\infty]$, and $q \in [1,\infty]$. Let $m \in \mathbb{Z}_+$ and $m > s$. Define

$$\|f\|^{(\mathrm{c})}_{B^s_{p,q}(\mathbb{R}^n);m} := \|f\|_{L^p(\mathbb{R}^n)} + \|t \in]0,1[\mapsto t^{-s-\frac{1}{q}} \omega^m_p(f;t)\|_{L^q(]0,1[)}$$

for all $f \in \mathrm{bor}(\mathbb{R}^n, \mathbb{C})$.

Norm $\|\cdot\|^{(\mathrm{c})}_{B^s_{p,q}(\mathbb{R}^n);m}$ is an equivalent norm on $B^s_{p,q}(\mathbb{R}^n)$ for the range of parameters given in Definition 2.14. Norm $\|\cdot\|^{(\mathrm{c})}_{B^s_{p,q}(\mathbb{R}^n);m}$ characterizes the Besov space $B^s_{p,q}(\mathbb{R}^n)$ on $\mathrm{bor}(\mathbb{R}^n, \mathbb{C})$ for the given range of parameters. Note that $L^p(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ for all $n \in \mathbb{Z}_+$ and $p \in [1,\infty]$. See Definition V.4.3 in Ref. 2.

2.5. *Tensor products*

Definition 2.15. When $n \in \mathbb{Z}_+$ define

$$\check{\mathbf{e}}^{\otimes}_{\mathbf{k}} := \bigotimes_{l=1}^{n} \check{\mathbf{e}}_{\mathbf{k}[l]}.$$

for all $\mathbf{k} \in \mathbb{Z}^n$.

When $n \in \mathbb{Z}_+$, α is a uniform crossnorm, and $A_1, \ldots, A_n$ are Banach spaces then the completed tensor product of several Banach spaces is defined by the recursive formula

$$A_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha A_n :=_{\text{n.s.}} (A_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha A_{n-1}) \hat{\otimes}_\alpha A_n.$$

The indexed completed tensor product is defined by

$$\hat{\bigotimes_{j=1}^{n}}_{\alpha} A_j :=_{\text{n.s.}} \left(\hat{\bigotimes_{j=1}^{n-1}}_{\alpha} A_j \right) \hat{\otimes}_\alpha A_n,$$

where $n > 1$. When $n = 1$ define

$$\hat{\bigotimes_{j=1}^{1}}_{\alpha} A_j :=_{\text{n.s.}} A_1.$$

Uniform crossnorms do not generally respect subspaces. However, when X is a closed subspace of a Banach space E, Y is a closed subspace F, and α is a reasonable crossnorm on $E \otimes F$ it is possible to use the norm inherited from $E \otimes_\alpha F$ on vector space $X \otimes Y$ (see Chap. 1 in Ref. 37). We give the following definition for this kind of construction.

Definition 2.16. Let E and F be Banach spaces and α a norm on $E \otimes F$. Let X be a closed subspace of E and Y a closed subspace of F. Define $X \otimes_{\alpha;E\hat{\otimes}_\alpha F} Y$ to be the normed vector space formed by using the norm inherited from $E\hat{\otimes}_\alpha F$ in the vector space $X \otimes Y$, i.e. $\|u\|_{X\otimes_{\alpha;E\hat{\otimes}_\alpha F}Y} := \alpha_{E,F}(u)$ for all $u \in X \otimes Y$. Define $X\hat{\otimes}_{\alpha;E\hat{\otimes}_\alpha F}Y$ to be the closure of $X \otimes_{\alpha;E\hat{\otimes}_\alpha F} Y$ in space $E\hat{\otimes}_\alpha F$. The notations $X \otimes_{\alpha;E\otimes F} Y$ and $X\hat{\otimes}_{\alpha;E\otimes \dot{F}}Y$ may also be used for the aforementioned definitions.

Now $X \otimes_{\alpha;E\hat{\otimes}_\alpha F} Y$ is a normed subspace of $E\hat{\otimes}_\alpha F$ and $X\hat{\otimes}_{\alpha;E\hat{\otimes}_\alpha F}Y$ is a closed subspace of $E\hat{\otimes}_\alpha F$.

Definition 2.17. Let $n \in \mathbb{Z}_+$. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be Banach spaces so that A_1 is a closed subspace of B_1 and $B_k \otimes A_{k+1}$ is a linear subspace of B_{k+1} for $k \in Z(n-1)$. When $k \geq 2$ define

$$\hat{\bigotimes_{j=1}^{k}}{}_{(B_j)} A_j :=_{\text{n.s.}} \operatorname{clos}_{B_k} T_k,$$

where

$$T_k :=_{\text{n.s.}} \left(\hat{\bigotimes_{j=1}^{k-1}}{}_{(B_j)} A_j \right) \otimes_{(B_k)} A_k.$$

When $k = 1$ define

$$\hat{\bigotimes_{j=1}^{k}}{}_{(B_j)} A_j :=_{\text{n.s.}} A_1.$$

When the assumptions of Definition 2.17 hold

$$\hat{\bigotimes_{j=1}^{k}}{}_{(B_j)} A_j$$

is a closed subspace of Banach space B_k for all $k \in Z(n)$.

Definition 2.18. Let $n \in \mathbb{Z}_+$. Let $A_1, \ldots, A_n$, $B_1, \ldots, B_n$, $E_1, \ldots, E_n$, and $F_1, \ldots, F_n$ be Banach spaces so that

- A_1 is a closed subspace of E_1 and $E_k \otimes A_{k+1}$ is a linear subspace of E_{k+1} for all $k = 1, \ldots, n-1$.
- B_1 is a closed subspace of F_1 and $F_k \otimes B_{k+1}$ is a linear subspace of F_{k+1} for all $k = 1, \ldots, n-1$.

Suppose that $P_k : A_k \to B_k$, $k = 1, \ldots, n$ are operators. Let $S_1 := P_1$, $T_1 := S_1 = P_1$, and $S_k := T_{k-1} \otimes P_k$ for $k = 2, \ldots, n$. When $k \in \{2, \ldots, n\}$ and S_k is continuous let

$$T_k : \hat{\bigotimes}_{l=1}^{k}{}_{(E_l)} A_l \to \hat{\bigotimes}_{l=1}^{k}{}_{(F_l)} B_l,$$

be the unique continuous linear extension of S_k to

$$\hat{\bigotimes}_{l=1}^{k}{}_{(E_l)} A_l.$$

If $k \in \{2, \ldots, n\}$ and S_k is not continuous let $T_k = 0$. When all of the functions S_k, T_k, $k = 1, \ldots, n$ are operators define

$$\bigotimes_{k=1}^{n}{}_{(E_k, F_k)} P_k = T_n.$$

If any of the functions S_k, T_k, $k = 1, \ldots, n$ is not an operator then

$$\bigotimes_{k=1}^{n}{}_{(E_k, F_k)} P_k$$

is undefined.

There is an isometric embedding

$$X^* \otimes_{\alpha^s} Y^* \subset_1 (X \otimes_\alpha Y)^* \tag{2.1}$$

for tensor norms α and Banach spaces X and Y,[41] see also Ref. 11. When $K = \mathbb{R}$ or $K = \mathbb{C}$, $n \in \mathbb{N} + 2$, and $X_1, \ldots, X_n$ are closed subspaces of $C_b(\mathbb{R}, K)$ the tensor product $X_1 \hat{\otimes}_\varepsilon \cdots \hat{\otimes}_\varepsilon X_n$ is a closed subspace of $C_b(\mathbb{R}^n, K)$.

3. General Definitions for a Compactly Supported Interpolating MRA

3.1. *Mother scaling function*

Definition 3.1. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. A compactly supported interpolating mother scaling function is a function $\varphi \in C_{\text{com}}(\mathbb{R}^n, K)$ satisfying the following conditions:

(MSF.1)

$$\forall \mathbf{k} \in \mathbb{Z}^n : \varphi(\mathbf{k}) = \delta_{\mathbf{k},0}$$

(MSF.2)

$$\forall \mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \varphi\left(\frac{\mathbf{k}}{2}\right) \varphi(2\mathbf{x} - \mathbf{k}).$$

The Deslauriers–Dubuc fundamental functions satisfy these conditions.[17]

3.2. *General definitions for the univariate MRA's*

We shall denote the function space for which the MRA is defined by E in this section. We have either $E = C_{\mathrm{u}}(\mathbb{R}, K)$ or $E = C_0(\mathbb{R}, K)$ where $K = \mathbb{R}$ or $K = \mathbb{C}$. We shall assume that $\varphi \in C_{\mathrm{com}}(\mathbb{R}, K)$ is a compactly supported interpolating mother scaling function throughout this subsection.

Definition 3.2. Define function $\psi \in C_{\mathrm{com}}(\mathbb{R}, K)$ by

$$\psi(x) := \varphi(2x - 1)$$

for all $x \in \mathbb{R}$. Function ψ is called the mother wavelet.

Definition 3.3. Define

$$\varphi_{j,k} := \varphi(2^j \cdot -k)$$

and

$$\psi_{j,k} := \psi(2^j \cdot -k)$$

for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

Definition 3.4. Define

$$\zeta_s := \begin{cases} \varphi; & s = 0, \\ \psi; & s = 1, \end{cases}$$

where $s \in \{0, 1\}$.

Definition 3.5. When $k \in \mathbb{Z}$ define

$$\begin{aligned} h_k &:= \varphi\left(\frac{k}{2}\right), \\ g_k &:= \delta_{k,1}, \\ \tilde{h}_k &:= \delta_{k,0}, \\ \tilde{g}_k &:= (-1)^{k-1} h_{1-k}. \end{aligned}$$

Definition 3.6. Define $\tilde{\varphi} := \delta \in E^*$ and $\tilde{\psi} \in E^*$ by

$$\tilde{\psi} := 2 \sum_{k \in \mathbb{Z}} \tilde{g}_k \tilde{\varphi}(2 \cdot -k). \tag{3.1}$$

Define $\tilde{\varphi}_{j,k} := 2^j \tilde{\varphi}(2^j \cdot -k)$ and $\tilde{\psi}_{j,k} := 2^j \tilde{\psi}(2^j \cdot -k)$ where $j, k \in \mathbb{Z}$.

As φ is compactly supported only a finite number of numbers h_k, $k \in \mathbb{Z}$, and the other three filters defined by Definition 3.5 are nonzero. Consequently the series in Eq. (3.1) has a finite number of nonzero terms.

Definition 3.7. Define

$$\tilde{\zeta}_s := \begin{cases} \tilde{\varphi}; & s = 0, \\ \tilde{\psi}; & s = 1, \end{cases}$$

where $s \in \{0, 1\}$.

3.3. *General definitions for multivariate MRA's*

We will assume that $n \in \mathbb{Z}_+$ and either $K = \mathbb{R}$ or $K = \mathbb{C}$ throughout this subsection. We will also assume that $E =_{\text{n.s.}} C_{\text{u}}(\mathbb{R}, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}, K)$. Furthermore, $\varphi \in C_{\text{com}}(\mathbb{R}, K)$ shall be a compactly supported interpolating mother scaling function throughout this subsection.

We set

$$F :=_{\text{n.s.}} \begin{cases} C_{\text{u}}(\mathbb{R}^n, K); & E = C_{\text{u}}(\mathbb{R}, K), \\ C_0(\mathbb{R}^n, K); & E = C_0(\mathbb{R}, K). \end{cases}$$

Definition 3.8. Define function $\varphi^{[n]} \in C_{\text{com}}(\mathbb{R}^n, K)$ by

$$\varphi^{[n]} := \bigotimes_{k=1}^{n} \varphi.$$

Function $\varphi^{[n]}$ is called an n-dimensional tensor product mother scaling function generated by φ. Define also $\varphi^{[n]}_{j,\mathbf{k}} := \varphi^{[n]}(2^j \cdot - \mathbf{k})$ where $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$.

Function $\varphi^{[n]}$ is a compactly supported interpolating mother scaling function on $\mathbb{R}^n$.

Definition 3.9. When $\mathbf{s} \in \{0,1\}^n$ define function $\psi^{[n]}_{\mathbf{s}} \in C_{\text{com}}(\mathbb{R}^n, K)$ by

$$\psi^{[n]}_{\mathbf{s}} := \bigotimes_{k=1}^{n} \zeta_{\mathbf{s}[k]}$$

and $\psi^{[n]}_{\mathbf{s},j,\mathbf{k}} := \psi^{[n]}_{\mathbf{s}}(2^j \cdot - \mathbf{k})$.

The domain of the Dirac δ functional varies in this paper. That is, we may keep δ as an element of different dual spaces A^*. When $z_1, \ldots, z_m \in \mathbb{R}$ we will identify $\delta(\cdot - z_1) \otimes \cdots \otimes \delta(\cdot - z_m)$ with $\delta(\cdot - (z_1, \ldots, z_m))$.

Definition 3.10. Define $\tilde{\varphi}^{[n]} \in F^*$ by

$$\tilde{\varphi}^{[n]} := \bigotimes_{l=1}^{n} \tilde{\varphi}$$

and $\tilde{\varphi}^{[n]}_{j,\mathbf{k}} \in F^*$ by $\tilde{\varphi}^{[n]}_{j,\mathbf{k}} := 2^j \tilde{\varphi}^{[n]}(2^j \cdot - \mathbf{k})$ where $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$. Define also $\tilde{\psi}^{[n]}_{\mathbf{s}} \in F^*$ by

$$\tilde{\psi}^{[n]}_{\mathbf{s}} := \bigotimes_{l=1}^{n} \tilde{\zeta}_{\mathbf{s}[l]},$$

where $\mathbf{s} \in \{0,1\}^n$ and $\tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}} \in F^*$ by $\tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}} := 2^j \tilde{\psi}^{[n]}_{\mathbf{s}}(2^j \cdot - \mathbf{k})$ where $\mathbf{s} \in \{0,1\}^n$, $j \in \mathbb{Z}$, and $\mathbf{k} \in \mathbb{Z}^n$.

Goedecker[20] gives formulas for wavelet filters, too.

Lemma 3.1. *Let $f \in \mathbb{C}$, $m \in \mathbb{N}$, $r_1 \in \mathbb{R}_+$, and $\tilde{f} \in C(\overline{B}_{\mathbb{R}}(0;r_1))^*$. Let*

$$f^{[n]} := \bigotimes_{i=0}^{n} f$$

and

$$\tilde{f}^{[n]} := \bigotimes_{i=0}^{n} \tilde{f}.$$

If $(\tilde{f}, f)$ spans all polynomials of degree at most m then $(\tilde{f}^{[n]}, f^{[n]})$ spans all polynomials of degree at most m.

Proof. Let

$$p(\mathbf{x}) := \prod_{i=1}^{n} p_i(\mathbf{x}[i]) = \prod_{i=1}^{n} (\mathbf{x}[i])^{\mathbf{a}[i]},$$

where $\mathbf{a}[i] \in Z_0(m)$, $i = 1, \ldots, n$. Define $r_2 \in \mathbb{R}_+$ so that $\left(\overline{B}_{\mathbb{R}}(0;r_1)\right)^n \subset \overline{B}_{\mathbb{R}^n}(0;r_2)$. Now

$$\begin{aligned}
&\sum_{\mathbf{k}\in\mathbb{Z}^n} \langle \tilde{f}^{[n]}(\cdot - \mathbf{k}), p|_{\overline{B}_{\mathbb{R}^n}(\mathbf{k};r_2)}\rangle f^{[n]}(\mathbf{x}-\mathbf{k}) \\
&\quad = \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_n\in\mathbb{Z}} \prod_{i=1}^{n} \langle \tilde{f}(\cdot - \mathbf{k}[i]), p_i|_{\overline{B}_{\mathbb{R}}(\mathbf{k}[i];r_1)}\rangle f(\mathbf{x}[i] - \mathbf{k}[i]) \\
&\quad = \prod_{i=1}^{n} \sum_{k\in\mathbb{Z}} \langle \tilde{f}(\cdot - k), p_i|_{\overline{B}_{\mathbb{R}}(k;r_1)}\rangle f(\mathbf{x}[i] - k) = \prod_{i=1}^{n} p_i(\mathbf{x}[i]) = p(\mathbf{x})
\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$. □

In particular, if φ spans all polynomials of degree at most m then $\varphi^{[n]}$ spans all polynomials of n variables and of degree at most m.

4. Compactly Supported Interpolating MRA of $C_{\mathrm{u}}(\mathbb{R}^n, K)$

We will assume that $n \in \mathbb{Z}_+$ and $K = \mathbb{R}$ or $K = \mathbb{C}$ throughout this section. We will also assume that $\varphi^{[n]} \in C_{\mathrm{com}}(\mathbb{R}^n, K)$ is an n-dimensional tensor product mother scaling function in this section. Unless otherwise stated, we shall assume that the same values of n, K, and $\varphi^{[n]}$ are used throughout this section. Chui and Li[6] have developed a MRA in the univariate case $C_{\mathrm{u}}(\mathbb{R})$.

Definition 4.1. Define

$$V_{n,j}^{(u)} := \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\varphi^{[n]}(2^j\mathbf{x} - \mathbf{k}) : \mathbf{a} \in l^\infty(\mathbb{Z}^n, K) \right\} \tag{4.1}$$

$$\|f|V_{n,j}^{(u)}\| := \|f\|_\infty, \quad f \in V_{n,j}^{(u)}$$

for all $j \in \mathbb{Z}$.

Definition 4.2. Let spaces $V_{n,j}^{(u)}$, $j \in \mathbb{Z}$, be defined by Definition 4.1. We call $\{V_{n,j}^{(u)} : j \in \mathbb{Z}\}$ an interpolating tensor product MRA of $C_{\mathrm{u}}(\mathbb{R}^n, K)$ generated by $\varphi^{[n]}$ provided that the following conditions are satisfied:

(MRA1.1) $\forall j \in \mathbb{Z} : V_{n,j}^{(u)} \subset V_{n,j+1}^{(u)}$
(MRA1.2) $\overline{\bigcup_{j\in\mathbb{Z}} V_{n,j}^{(u)}} = C_{\mathrm{u}}(\mathbb{R}^n, K)$
(MRA1.3) $\bigcap_{j\in\mathbb{Z}} V_{n,j}^{(u)} = K$
(MRA1.4) $\forall j \in \mathbb{Z}, f \in K^{\mathbb{R}^n} : f \in V_{n,j}^{(u)} \Leftrightarrow f(2\cdot) \in V_{n,j+1}^{(u)}$
(MRA1.5) $\forall j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n, f \in K^{\mathbb{R}^n} : f \in V_{n,j}^{(u)} \Leftrightarrow f(\cdot - 2^{-j}\mathbf{k}) \in V_{n,j}^{(u)}$
(MRA1.6) $\forall \mathbf{k} \in \mathbb{Z}^n : \varphi^{[n]}(\mathbf{k}) = \delta_{\mathbf{k},0}$.

Our requirements for the definition of interpolating multiresolution analysis are stricter and simpler than those in Ref. 6. Condition (MRA1.6) is replaced by a weaker condition for φ in Ref. 6 but it is possible to construct function φ_L that satisfies condition (MRA1.6) and generates the same subspaces $V_{1,j}^{(u)}$ as function φ.

Under the conditions given at the start of this section ($\varphi^{[n]}$ is an n-dimensional tensor product mother scaling function) the conditions (MRA1.1), (MRA1.4), (MRA1.5) and (MRA1.6) are true. We will show later that conditions (MRA1.2) and (MRA1.3) are true, too.

Spaces $V_{n,j}^{(u)}$, $j \in \mathbb{Z}$, are topologically isomorphic to $l^\infty(\mathbb{Z}^n, K)$. It has been proved in Ref. 28 that

$$\sum_{\mathbf{k}\in\mathbb{Z}^n} \varphi^{[n]}(\mathbf{y} - \mathbf{k}) = \widehat{\varphi^{[n]}}(0)$$

for all $\mathbf{y} \in \mathbb{R}^n$. It follows from the cardinal interpolation property (MSF.1) that

$$\sum_{\mathbf{k}\in\mathbb{Z}^n} \varphi^{[n]}(\mathbf{y} - \mathbf{k}) = 1 \tag{4.2}$$

for all $\mathbf{y} \in \mathbb{R}^n$.

Remark 4.1. The series in Eq. (4.1) converges (pointwise) for all $\mathbf{a} \in l^\infty(\mathbb{Z}^n, K)$ but the series $\sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\varphi^{[n]}(2^j \cdot - \mathbf{k})$ need not converge in the norm topology of $C_{\mathrm{u}}(\mathbb{R}^n, K)$. For example, consider sequence $\mathbf{a}[\mathbf{k}] = 1$, $\mathbf{k} \in \mathbb{Z}^n$. Now $\sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\varphi^{[n]}(2^j\mathbf{x} - \mathbf{k}) = 1$ for all $\mathbf{x} \in \mathbb{R}^n$ but the series $\sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\varphi^{[n]}(2^j \cdot - \mathbf{k})$ does not converge in the norm topology of $C_{\mathrm{u}}(\mathbb{R}^n, K)$.

Theorem 4.1. *We have*

$$\bigcap_{j\in\mathbb{Z}} V_{n,j}^{(u)} = K.$$

Proof. The proof is similar to a part of the proof of [6, Theorem 3.2]. □

Definition 4.3. When $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$ define

$$W^{(\mathrm{u})}_{n,\mathbf{s},j} := \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \psi^{[n]}_{\mathbf{s},j,\mathbf{k}}(\mathbf{x}) : \mathbf{a} \in l^\infty(\mathbb{Z}^n, K) \right\}$$
$$\|f|W^{(\mathrm{u})}_{n,\mathbf{s},j}\| := \|f\|_\infty, \quad f \in W^{(\mathrm{u})}_{n,\mathbf{s},j}.$$

Spaces $W^{(\mathrm{u})}_{n,\mathbf{s},j}$, $\mathbf{s} \in \{0,1\}^n$, $j \in \mathbb{Z}$, are topologically isomorphic to $l^\infty(\mathbb{Z}^n, K)$.

Definition 4.4. When $\mathbf{s} \in \{0,1\}^n$ and $j \in \mathbb{Z}$ define

$$(Q^{(\mathrm{u})}_{n,\mathbf{s},j} f)(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}, f \rangle \psi^{[n]}_{\mathbf{s},j,\mathbf{k}}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $f \in C_\mathrm{u}(\mathbb{R}^n, K)$. When $j \in \mathbb{Z}$ define $P^{(\mathrm{u})}_{n,j} := Q^{(\mathrm{u})}_{n,\mathbf{0}_n,j}$.

Operator $Q^{(\mathrm{u})}_{n,\mathbf{s},j}$ is a continuous projection of $C_\mathrm{u}(\mathbb{R}^n, K)$ onto $W^{(\mathrm{u})}_{n,\mathbf{s},j}$ for each $\mathbf{s} \in \{0,1\}^n$ and $j \in \mathbb{Z}$.

Definition 4.5. When $j \in \mathbb{Z}$ define

$$W^{(\mathrm{u})}_{n,j} :=_{\mathrm{n.s.}} \dot{\sum_{\mathbf{s} \in J_+(n)}} W^{(\mathrm{u})}_{n,\mathbf{s},j}$$

and

$$Q^{(\mathrm{u})}_{n,j} := \sum_{\mathbf{s} \in J_+(n)} Q^{(\mathrm{u})}_{n,\mathbf{s},j}.$$

Spaces $W^{(\mathrm{u})}_{n,j}$, $j \in \mathbb{Z}$, are topologically isomorphic to $l^\infty(J_+(n) \times \mathbb{Z}^n, K)$. When $j \in \mathbb{Z}$ we have

$$V^{(u)}_{n,j+1} =_{\mathrm{n.s.}} V^{(u)}_{n,j} \dot{+} W^{(\mathrm{u})}_{n,j} =_{\mathrm{n.s.}} V^{(u)}_{n,j} \dot{+} \dot{\sum_{\mathbf{s} \in J_+(n)}} W^{(\mathrm{u})}_{n,\mathbf{s},j} =_{\mathrm{n.s.}} \dot{\sum_{\mathbf{s} \in \{0,1\}^n}} W^{(\mathrm{u})}_{n,\mathbf{s},j}. \tag{4.3}$$

Operator $Q^{(\mathrm{u})}_{n,j}$ is a continuous projection of $C_\mathrm{u}(\mathbb{R}^n, K)$ onto $W^{(\mathrm{u})}_{n,j}$ for each $j \in \mathbb{Z}$. We also have $Q^{(\mathrm{u})}_{n,j} = P^{(\mathrm{u})}_{n,j+1} - P^{(\mathrm{u})}_{n,j}$ for all $j \in \mathbb{Z}$.

Theorem 4.2. *There exist $c_1 \in \mathbb{R}_+$ and $c_2 \in \mathbb{R}_+$ so that*

$$\|f - P^{(\mathrm{u})}_{n,j} f\|_\infty \leq c_1 \omega(f; 2^{-j} c_2)$$

for all $f \in C_\mathrm{u}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$.

Proof. Let $f \in C_\mathrm{u}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$. Define $r_1 := r_{\mathrm{supp}}(\varphi^{[n]})$. Let $\mathbf{x} \in \mathbb{R}^n$. Define $I := I_{\mathrm{trans}}(\varphi^{[n]}, 2^j \mathbf{x})$. Now by Eq. (4.2) we have

$$\begin{aligned}(f - P^{(\mathrm{u})}_{n,j} f)(\mathbf{x}) &= \sum_{\mathbf{k} \in \mathbb{Z}^n} \left(f(\mathbf{x}) - f\left(\frac{\mathbf{k}}{2^j}\right)\right) \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) \\ &= \sum_{\mathbf{k} \in I} \left(f(\mathbf{x}) - f\left(\frac{\mathbf{k}}{2^j}\right)\right) \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}).\end{aligned}$$

Let $\ell \in I$. Now $2^j\mathbf{x} - \ell \in \overline{B}_{\mathbb{R}^n}(0; r_1)$ from which it follows that $\mathbf{x} - 2^{-j}\ell \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j}r_1)$. Hence

$$\left| f(\mathbf{x}) - f\left(\frac{\ell}{2^j}\right)\right| \leq \omega(f; 2^{-j}r_1). \tag{4.4}$$

Let $c_1 := N_{\mathrm{cover}}(\varphi^{[n]})\|\varphi^{[n]}\|_\infty$ and $c_2 := r_1 = r_{\mathrm{supp}}(\varphi^{[n]})$. By Eq. (4.4) we have

$$|(f - P_{n,j}^{(\mathrm{u})} f)(\mathbf{x})| \leq c_1 \omega(f; 2^{-j}c_2). \qquad \square$$

Theorem 4.3. *Let $f \in C_{\mathrm{u}}(\mathbb{R}^n)$. Now*

$$\lim_{j\to\infty} \|f - P_{n,j}^{(\mathrm{u})} f\|_\infty = 0.$$

Proof. See also Theorem 3.2 in Ref. 6 and Theorem 2.4 in Ref. 17. Let $j \in \mathbb{Z}$. By Theorem 4.2 we have $\|f - P_{n,j}^{(\mathrm{u})} f\|_\infty \leq c_1\omega(f; 2^{-j}c_2)$ where c_1 and c_2 do not depend on j or f. Since f is uniformly continuous

$$\lim_{t\to 0} \omega(f; t) = 0$$

and hence

$$\lim_{j\to\infty} \|f - P_{n,j}^{(\mathrm{u})} f\|_\infty = 0. \qquad \square$$

Corollary 4.1.

$$\overline{\bigcup_{j\in\mathbb{Z}} V_{n,j}^{(u)}} = C_{\mathrm{u}}(\mathbb{R}^n, K).$$

Theorem 4.4. *Let $t \in \mathbb{R}_+$. There exists a constant $c_1 \in \mathbb{R}_+$, which may depend on t, so that*

$$\forall\, f \in C_{\mathrm{u}}(\mathbb{R}^n), \quad j \in \mathbb{Z} : \omega(P_{n,j}^{(\mathrm{u})} f; 2^{-j}t) \leq c_1\omega(f; 2^{-j}\sqrt{n}).$$

Proof. The proof of case $n = 1$ is similar to the proof in Sec. 7.1 in Ref. 17. We will assume that $n \geq 2$ in the sequel. Define bijection $\sigma_{\mathrm{c}}^{[n]} : \mathbb{N} \twoheadrightarrow \mathbb{Z}^n$ so that $\|\sigma_{\mathrm{c}}^{[n]}(p+1) - \sigma_{\mathrm{c}}^{[n]}(p)\|_\infty = 1$ for all $p \in \mathbb{N}$. It can be proved that this kind of bijection always exists.[26] Define

$$\begin{aligned} v_{j,l}(\mathbf{x}, \mathbf{h}) := \sum_{p=0}^{l} (&\varphi^{[n]}(2^j\mathbf{x} - \sigma_{\mathrm{c}}^{[n]}(p) - \lfloor 2^j\mathbf{x}\rfloor + 2^j\mathbf{h}) \\ &- \varphi^{[n]}(2^j\mathbf{x} - \sigma_{\mathrm{c}}^{[n]}(p) - \lfloor 2^j\mathbf{x}\rfloor)), \end{aligned}$$

where $j, l \in \mathbb{N}$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j}t)$. Using Eq. (4.2) it can be proved that there exists $c_1 \in \mathbb{R}_+$ so that

$$\sum_{l=0}^{\infty} |v_{j,l}(\mathbf{x}, \mathbf{h})| \leq c_1$$

for all $j \in \mathbb{N}$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j}t)$. Let $f \in C_{\mathrm{u}}(\mathbb{R}^n)$, $j_1 \in \mathbb{Z}$, $\mathbf{x}_1 \in \mathbb{R}^n$, and $\mathbf{h}_1 \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j}t)$. Let $\mathbf{a} := (f(2^{-j_1}\mathbf{k}))_{\mathbf{k}\in\mathbb{Z}^n}$. Now

$$\begin{aligned} z &:= (P_{n,j_1}^{(\mathrm{u})} f)(\mathbf{x}_1 + \mathbf{h}_1) - (P_{n,j_1}^{(\mathrm{u})} f)(\mathbf{x}_1) \\ &= \sum_{l=0}^{\infty} (\mathbf{a}[\sigma_{\mathrm{c}}^{[n]}(l)] - \mathbf{a}[\sigma_{\mathrm{c}}^{[n]}(l+1)]) v_{j_1,l}(\mathbf{x}_1, \mathbf{h}_1) \end{aligned}$$

from which it follows that

$$|z| \leq \left(\sup_{l\in\mathbb{N}} |\mathbf{a}[\sigma_{\mathrm{c}}^{[n]}(l)] - \mathbf{a}[\sigma_{\mathrm{c}}^{[n]}(l+1)]| \right) \cdot \sum_{l=0}^{\infty} |v_{j_1,l}(\mathbf{x}_1, \mathbf{h}_1)| \leq \omega(f; 2^{-j}\sqrt{n}) \cdot c_1. \qquad \square$$

Theorem 4.5. *Let $j_0 \in \mathbb{Z}$. Suppose that the mother scaling function φ is Lipschitz continuous and let $\varphi^{[n]}$ be the tensor product mother scaling function generated by φ. Then*

$$V_{n,j_0}^{(u)} \dot{+} \dot{\sum_{j=j_0}^{\infty}} W_{n,j}^{(\mathrm{u})} \neq C_{\mathrm{u}}(\mathbb{R}^n, K).$$

Proof. Let

$$A := V_{n,j_0}^{(u)} \dot{+} \dot{\sum_{j=j_0}^{\infty}} W_{n,j}^{(\mathrm{u})}.$$

Define function $f \in C_{\mathrm{u}}(\mathbb{R}, K)$ by

$$f(x) := \begin{cases} \sqrt{x}; & x \in [0,1], \\ -x+2; & x \in [1,2], \\ 0; & x \leq 0 \vee x \geq 2 \end{cases}$$

and function $f^{[n]} \in C_{\mathrm{u}}(\mathbb{R}^n, K)$ by

$$f^{[n]} := \bigotimes_{k=1}^{n} f.$$

Functions f and $f^{[n]}$ are not Lipschitz continuous. As φ is Lipschitz continuous all the functions $P_{n,j}^{(\mathrm{u})} g$ are Lipschitz continuous for each $g \in C_{\mathrm{u}}(\mathbb{R}^n, K)$. A is a locally convex space with an inductive limit topology. It follows from Theorem 6.2 in Ref. 42 that the space A is complete.

Suppose that A would be equal to $C_{\mathrm{u}}(\mathbb{R}^n, K)$ as a set. Then we would have $f^{[n]} \in A$. It follows from the definition of the locally convex direct sum that

$$f^{[n]} \in V_{n,j_0}^{(u)} \dot{+} \dot{\sum_{j=j_0}^{j_1}} W_{n,j}^{(\mathrm{u})}$$

for some $j_1 \in \mathbb{Z}$, $j_1 \geq j_0$. Now $f^{[n]} = P_{n,j_1+1}^{(\mathrm{u})} f^{[n]}$. It follows that $f^{[n]}$ is Lipschitz continuous, which is a contradiction. Hence A is not equal to $C_{\mathrm{u}}(\mathbb{R}^n, K)$ as a set. $\square$

The definition of locally convex direct sums from Ref. 42 is used here. An example of a Lipschitz continuous mother scaling function is a Deslauriers–Dubuc fundamental function with Hölder regularity greater than 1. By Eq. (4.3) and Corollary 4.1 we have

$$C_{\mathrm{u}}(\mathbb{R}^n, K) =_{\text{n.s.}} \operatorname{clos}\left(\bigcup_{l=j_0}^{\infty}\left(V_{n,j_0}^{(u)} \dot{+} \sum_{j=j_0}^{l} W_{n,j}^{(\mathrm{u})}\right)\right)$$

for all $j_0 \in \mathbb{Z}$.

5. Compactly Supported Interpolating MRA of $C_0(\mathbb{R}^n, K)$

A MRA of space $C_0(\mathbb{R}^n, K)$ can be developed in the same way as the MRA of space $C_{\mathrm{u}}(\mathbb{R}^n, K)$. Donoho[17] has developed a MRA for the one-dimensional case $C_0(\mathbb{R})$. We will assume that $n \in \mathbb{Z}_+$ and $K = \mathbb{R}$ or $K = \mathbb{C}$ throughout this section. We will also assume that $\varphi^{[n]} \in C_{\mathrm{com}}(\mathbb{R}^n, K)$ is an n-dimensional tensor product mother scaling function in this section. Unless otherwise stated, we shall assume that the same values of n, K, and $\varphi^{[n]}$ are used throughout this section. Note that many of the series that are pointwise convergent with $C_{\mathrm{u}}(\mathbb{R}^n, K)$ converge in the norm in $C_0(\mathbb{R}^n, K)$. We are also able to represent many subspaces and operators related to the MRA of $C_0(\mathbb{R}^n, K)$ as tensor products of the corresponding one-dimensional cases.

Definition 5.1. Define

$$V_{n,j}^{(0)} := \left\{\sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\varphi^{[n]}(2^j \cdot -\mathbf{k}) : \mathbf{a} \in c_0(\mathbb{Z}^n, K)\right\}$$
$$\|f|V_{n,j}^{(0)}\| := \|f\|_\infty, \quad f \in V_{n,j}^{(0)} \tag{5.1}$$

for each $j \in \mathbb{Z}$.

Definition 5.2. Let spaces $V_{n,j}^{(0)}$, $j \in \mathbb{Z}$, be defined by Definition 5.1. We call $\{V_{n,j}^{(0)} : j \in \mathbb{Z}\}$ an interpolating tensor product MRA of $C_0(\mathbb{R}^n, K)$ generated by $\varphi^{[n]}$ provided that the following conditions are satisfied:

(MRA2.1) $\forall j \in \mathbb{Z} : V_{n,j}^{(0)} \subset V_{n,j+1}^{(0)}$

(MRA2.2) $\overline{\bigcup_{j\in\mathbb{Z}} V_{n,j}^{(0)}} = C_0(\mathbb{R}^n, K)$

(MRA2.3) $\bigcap_{j\in\mathbb{Z}} V_{n,j}^{(0)} = \{0\}$

(MRA2.4) $\forall\, j \in \mathbb{Z}, f \in K^{\mathbb{R}^n} : f \in V_{n,j}^{(0)} \Leftrightarrow f(2\cdot) \in V_{n,j+1}^{(0)}$

(MRA2.5) $\forall\, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n, f \in K^{\mathbb{R}^n} : f \in V_{n,j}^{(0)} \Leftrightarrow f(\cdot - 2^{-j}\mathbf{k}) \in V_{n,j}^{(0)}$

(MRA2.6) $\forall\, \mathbf{k} \in \mathbb{Z}^n : \varphi^{[n]}(\mathbf{k}) = \delta_{\mathbf{k},0}$

Note that in this definition the intersection of spaces $V_{n,j}^{(0)}$ is $\{0\}$ instead of $\mathbb{C}$ (all complex valued constant functions on $\mathbb{R}^n$) as in Ref. 6 since here the MRA is constructed for functions vanishing at infinity. Donoho[17] does not require the mother scaling function φ to be compactly supported but φ has to be of rapid decay in his construction. He also includes requirements for the regularity and polynomial span of φ, which are needed for the norm equivalences to Besov and Triebel–Lizorkin spaces, into the definition of the MRA. Under the conditions given at the start of this section ($\varphi^{[n]}$ is an n-dimensional tensor product mother scaling function) the conditions (MRA2.1), (MRA2.2), (MRA2.3), (MRA2.4), (MRA2.5) and (MRA2.6) are true.

Definition 5.3. Let $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$. Define

$$W_{n,\mathbf{s},j}^{(0)} := \left\{ \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\psi_{\mathbf{s},j,\mathbf{k}}^{[n]} : \mathbf{a} \in c_0(\mathbb{Z}^n, K) \right\}$$

$$\|f|W_{n,\mathbf{s},j}^{(0)}\| := \|f\|_\infty, \quad f \in W_{n,\mathbf{s},j}^{(0)}.$$

Definition 5.4. Let $j \in \mathbb{Z}$. Define operator $P_{n,j}^{(0)} : C_0(\mathbb{R}^n, K) \to V_{n,j}^{(0)}$ by

$$P_{n,j}^{(0)} f = \sum_{\mathbf{k}\in\mathbb{Z}^n} \langle \tilde{\varphi}_{j,\mathbf{k}}^{[n]}, f\rangle \varphi_{j,\mathbf{k}}^{[n]}$$

for all $f \in C_0(\mathbb{R}^n, K)$. Define operator $Q_{n,\mathbf{s},j}^{(0)} : C_0(\mathbb{R}^n, K) \to W_{n,\mathbf{s},j}^{(0)}$ by

$$Q_{n,\mathbf{s},j}^{(0)} f = \sum_{\mathbf{k}\in\mathbb{Z}^n} \langle \tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]}, f\rangle \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}$$

for all $f \in C_0(\mathbb{R}^n, K)$ and $\mathbf{s} \in \{0,1\}^n$.

We now have

$$V_{n,j}^{(0)} =_{\text{n.s.}} \hat{\bigotimes_{k=1}^{n}}_{\varepsilon} V_{1,j}^{(0)}, \quad W_{n,\mathbf{s},j}^{(0)} =_{\text{n.s.}} \hat{\bigotimes_{k=1}^{n}}_{\varepsilon} W_{1,\mathbf{s}[k],j}^{(0)},$$

$$P_{n,j}^{(0)} = \bigotimes_{k=1}^{n}{}_{\varepsilon} P_{1,j}^{(0)}, \quad \text{and} \quad Q_{n,\mathbf{s},j}^{(0)} = \bigotimes_{k=1}^{n}{}_{\varepsilon} Q_{1,\mathbf{s}[k],j}^{(0)}.$$

We give now a general result on the tensor products of the function space $C_0(\mathbb{R}, K)$ with itself.

Theorem 5.1. *We have*

$$\hat{\bigotimes_{j=1}^{n}}_{\varepsilon} C_0(\mathbb{R}, K) =_{\text{n.s.}} C_0(\mathbb{R}^n, K).$$

Proof. Define

$$F^{[n]} :=_{\text{n.s.}} \hat{\bigotimes_{j=1}^{n}}_{\varepsilon} C_0(\mathbb{R}, K).$$

We have $F^{[n]} \subset C_0(\mathbb{R}^n, K)$. Let φ be the Deslauriers–Dubuc fundamental function of some degree m and $\varphi^{[n]}$ the n-dimensional tensor product mother scaling function generated by φ. Let spaces $V_{1,j}^{(0)}$, $j \in \mathbb{Z}$, belong to the interpolating multiresolution analysis generated by φ and spaces $V_{n,j}^{(0)}$, $j \in \mathbb{Z}$, be the corresponding tensor product spaces. Let $f \in C_0(\mathbb{R}^n, K)$. It follows from Theorem 4.3 that $P_{n,j}^{(0)} f \to f$ as $j \to \infty$. Now $P_{n,j}^{(0)} f \in V_{n,j}^{(0)} \subset F^{[n]}$ for all $j \in \mathbb{Z}$ from which it follows that $f \in \overline{F^{[n]}} = F^{[n]}$. Consequently $F^{[n]} =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. □

For example, $C_0(\mathbb{R}) \hat{\otimes}_\varepsilon C_0(\mathbb{R}) =_{\text{n.s.}} C_0(\mathbb{R}^2)$. As in the case of $C_{\text{u}}(\mathbb{R}^n, K)$ we get

$$V_{n,j_0}^{(0)} \dotplus \dot{\sum_{j=j_0}^{\infty}} W_{n,j}^{(0)} \neq C_0(\mathbb{R}^n, K)$$

when the mother scaling function is Lipschitz continuous. We also have

$$C_0(\mathbb{R}^n, K) =_{\text{n.s.}} \operatorname{clos}\left(\bigcup_{l=j_0}^{\infty}\left(V_{n,j_0}^{(0)} \dotplus \dot{\sum_{j=j_0}^{l}} W_{n,j}^{(0)}\right)\right)$$

for all $j_0 \in \mathbb{Z}$.

6. Interpolating Dual MRA

We shall have $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$ throughout this section. We shall also have $E = C_{\text{u}}(\mathbb{R}^n, K)$ or $E = C_0(\mathbb{R}^n, K)$.

6.1. *General*

Definition 6.1. When $j \in \mathbb{Z}$ define

$$\tilde{V}_{n,j} :=_{\text{n.s.}} \left\{\sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{d}[\mathbf{k}]\tilde{\varphi}_{j,\mathbf{k}}^{[n]} : \mathbf{d} \in l^1(\mathbb{Z}^n, K)\right\}$$

$$\|\tilde{f}|\tilde{V}_{n,j}\| := \|\tilde{f}|C_{\text{u}}(\mathbb{R}^n, K)^*\| = \|\tilde{f}|C_0(\mathbb{R}^n, K)^*\|, \quad \tilde{f} \in \tilde{V}_{n,j}.$$

We may identify Banach space $\tilde{V}_{n,j}$ for both $E = C_{\text{u}}(\mathbb{R}^n, K)$ and $E = C_0(\mathbb{R}^n, K)$ for each $j \in \mathbb{Z}$. It follows also that function $\tilde{\iota}_{n,j} : l^1(\mathbb{Z}^n, K) \to \tilde{V}_{n,j}$ defined by

$$\tilde{\iota}_{n,j}(\mathbf{d}) := \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{d}[\mathbf{k}]\tilde{\varphi}_{j,\mathbf{k}}^{[n]}$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$ is an isometric isomorphism from $l^1(\mathbb{Z}^n, K)$ onto $\tilde{V}_{n,j}$. We have

$$\tilde{f} \in \tilde{V}_{n,j} \Leftrightarrow \tilde{f}(2\cdot) \in \tilde{V}_{n,j+1}$$

and

$$\tilde{f} \in \tilde{V}_{n,j} \Leftrightarrow \tilde{f}(\cdot - 2^{-j}\mathbf{k}) \in \tilde{V}_{n,j}.$$

for all $\tilde{f} \in C_{\text{u}}(\mathbb{R}^n, K)^*$, $j \in \mathbb{Z}$, and $\mathbf{k} \in \mathbb{Z}^n$.

Definition 6.2. When $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$ define

$$\tilde{W}_{n,\mathbf{s},j} :=_{\text{n.s.}} \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}} : \mathbf{d} \in l^1(\mathbb{Z}^n, K) \right\},$$

$$\|\tilde{f} | \tilde{W}_{n,\mathbf{s},j}\| := \|\tilde{f} | C_{\text{u}}(\mathbb{R}^n, K)^*\| = \|\tilde{f} | C_0(\mathbb{R}^n, K)^*\|, \quad \tilde{f} \in \tilde{W}_{n,\mathbf{s},j}.$$

Spaces $\tilde{W}_{n,\mathbf{s},j}$, $\mathbf{s} \in \{0,1\}^n$, $j \in \mathbb{Z}$, are topologically isomorphic to $l^1(\mathbb{Z}^n, K)$. Assume that $j \in \mathbb{Z}$, $\mathbf{d} \in l^1(J_+(n) \times \mathbb{Z}^n, K)$, and

$$\tilde{f} := \sum_{\mathbf{s} \in J_+(n)} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{s}, \mathbf{k}] \tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}.$$

Now $\|\tilde{f} | C_{\text{u}}(\mathbb{R}^n, K)^*\| = \|\tilde{f} | C_0(\mathbb{R}^n, K)^*\|$. Consequently we may identify the Banach space $\tilde{W}_{n,\mathbf{s},j}$ for both $E = C_{\text{u}}(\mathbb{R}^n, K)$ and $E = C_0(\mathbb{R}^n, K)$ for each $\mathbf{s} \in \{0,1\}^n$ and $j \in \mathbb{Z}$.

Definition 6.3. When $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$ define

$$\tilde{P}_{n,j} \tilde{f} := \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{f}, \varphi^{[n]}_{j,\mathbf{k}} \rangle \tilde{\varphi}^{[n]}_{j,\mathbf{k}}$$

for all $\tilde{f} \in E^*$.

Definition 6.4. When $n \in \mathbb{Z}_+$, $\mathbf{s} \in \{0,1\}^n$, and $j \in \mathbb{Z}$ define

$$\tilde{Q}_{n,\mathbf{s},j} \tilde{f} := \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{f}, \psi^{[n]}_{\mathbf{s},j,\mathbf{k}} \rangle \tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}$$

for all $\tilde{f} \in E^*$.

We have $\tilde{Q}_{n,\mathbf{0}_n,j} = \tilde{P}_{n,j}$ for all $j \in \mathbb{Z}$.

Lemma 6.1. (i) $\forall j \in \mathbb{Z}, \mathbf{s} \in \{0,1\}^n, \mathbf{t} \in \{0,1\}^n : \mathbf{s} \neq \mathbf{t} \Rightarrow \forall \tilde{f} \in \tilde{W}_{n,\mathbf{s},j} : \tilde{Q}_{n,\mathbf{t},j} \tilde{f} = 0$.

(ii) *Operator $\tilde{Q}_{n,\mathbf{s},j}$ is a continuous projection of E^* onto $\tilde{W}_{n,\mathbf{s},j}$ for all $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$.*

(iii) *$\tilde{Q}_{n,\mathbf{s},j} = \tilde{Q}_{n,\mathbf{s},j} \circ \tilde{P}_{n,j'}$ for all $j, j' \in \mathbb{Z}$, $j < j'$, and $\mathbf{s} \in \{0,1\}^n$.*

For example, when $n = 2$ and $\mathbf{t} = (0,1)$ we have $\tilde{Q}_{n,\mathbf{t},j} \delta = 0$ for all $j \in \mathbb{Z}$. When $j \in \mathbb{Z}$, $\mathbf{s}, \mathbf{t} \in \{0,1\}^n$, and $\mathbf{s} \neq \mathbf{t}$ we have $\tilde{W}_{n,\mathbf{s},j} \cap \tilde{W}_{n,\mathbf{t},j} = \{0\}$.

Definition 6.5. When $j \in \mathbb{Z}$ define

$$\tilde{W}_{n,j} :=_{\text{n.s.}} \dot{\sum_{\mathbf{s} \in J_+(n)}} \tilde{W}_{n,\mathbf{s},j}.$$

Definition 6.6. When $j \in \mathbb{Z}$ define

$$\tilde{Q}_{n,j} := \sum_{\mathbf{s} \in J_+(n)} \tilde{Q}_{n,\mathbf{s},j}.$$

Assume that $j \in \mathbb{Z}$. Now operator $\tilde{Q}_{n,j}$ is a continuous projection of E^* onto $\tilde{W}_{n,j}$ and $\tilde{Q}_{n,j}\tilde{f} = 0$ for all $\tilde{f} \in \tilde{V}_{n,j}$. We also have $\tilde{Q}_{n,j} = \tilde{P}_{n,j+1} - \tilde{P}_{n,j}$ and $\tilde{V}_{n,j+1} =_{\text{n.s.}} \tilde{V}_{n,j} \dot{+} \tilde{W}_{n,j}$.

Theorem 6.1. (i) $\forall j \in \mathbb{Z}, \mathbf{s} \in \{0,1\}^n, \mathbf{t} \in \{0,1\}^n : \mathbf{s} \neq \mathbf{t} \Leftrightarrow \tilde{W}_{n,\mathbf{s},j} \perp W^{(\mathrm{u})}_{n,\mathbf{t},j}$,

(ii) $\forall j_1, j_2 \in \mathbb{Z} : j_1 \leq j_2 \Leftrightarrow \tilde{V}_{n,j_1} \perp W^{(\mathrm{u})}_{n,j_2}$,

(iii) $\forall j_1, j_2 \in \mathbb{Z} : j_1 \geq j_2 \Leftrightarrow \tilde{W}_{n,j_1} \perp V^{(u)}_{n,j_2}$,

(iv) $\forall j_1, j_2 \in \mathbb{Z} : j_1 \neq j_2 \Leftrightarrow \tilde{W}_{n,j_1} \perp W^{(\mathrm{u})}_{n,j_2}$.

Proof. Let $j \in \mathbb{Z}$, $\mathbf{s}, \mathbf{t} \in \{0,1\}^n$, and $\mathbf{k}, \ell \in \mathbb{Z}^n$. Proposition (i) follows from $\langle \tilde{\psi}^{[n]}_{\mathbf{s},j,\ell}, \psi^{[n]}_{\mathbf{t},j,\mathbf{k}} \rangle = \delta_{\mathbf{s},\mathbf{t}} \delta_{\ell,\mathbf{k}}$. Suppose that $j_1, j_2 \in \mathbb{Z}$ and $j_1 \neq j_2$. If $j_1 > j_2$ then $W^{(\mathrm{u})}_{n,j_2} \subset V^{(u)}_{n,j_1}$ and $\tilde{W}_{n,j_1} \perp V^{(u)}_{n,j_1}$ and hence $\tilde{W}_{n,j_1} \perp W^{(\mathrm{u})}_{n,j_2}$. If $j_1 < j_2$ then $\tilde{W}_{n,j_1} \subset \tilde{V}_{n,j_2}$ and $\tilde{V}_{n,j_2} \perp W^{(\mathrm{u})}_{n,j_2}$ and hence $\tilde{W}_{n,j_1} \perp W^{(\mathrm{u})}_{n,j_2}$. So proposition (iv) is true. □

Theorem 6.2. *Let $\tilde{f} \in C_0(\mathbb{R}^n, K)^*$. Then $\tilde{P}_{n,j}\tilde{f} \to \tilde{f}$ in the weak-* topology of $C_0(\mathbb{R}^n, K)^*$ as $j \to \infty$.*

Proof. Let $\tilde{f} \in C_0(\mathbb{R}^n, K)^*$. Suppose that $f \in C_0(\mathbb{R}^n, K)$. Then

$$\begin{aligned} \langle \tilde{P}_{n,j}\tilde{f}, P^{(0)}_{n,j} f \rangle &= \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{f}, \varphi^{[n]}_{j,\mathbf{k}} \rangle \langle \tilde{\varphi}^{[n]}_{j,\mathbf{k}}, P^{(0)}_{n,j} f \rangle \\ &= \left\langle \tilde{f}, \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{\varphi}^{[n]}_{j,\mathbf{k}}, f \rangle \varphi^{[n]}_{j,\mathbf{k}} \right\rangle \\ &= \langle \tilde{f}, P^{(0)}_{n,j} f \rangle. \end{aligned}$$

There exists $c \in \mathbb{R}_+$ so that $\|\tilde{P}_{n,j}\| \leq c$ for all $j \in \mathbb{Z}$. Consequently

$$\begin{aligned} |\langle \tilde{f}, f \rangle - \langle \tilde{P}_{n,j}\tilde{f}, f \rangle| &\leq |\langle \tilde{f}, f \rangle - \langle \tilde{f}, P^{(0)}_{n,j} f \rangle| \\ &\quad + |\langle \tilde{P}_{n,j}\tilde{f}, P^{(0)}_{n,j} f \rangle - \langle \tilde{P}_{n,j}\tilde{f}, f \rangle| \\ &= |\langle \tilde{f}, f - P^{(0)}_{n,j} f \rangle| + |\langle \tilde{P}_{n,j}\tilde{f}, P^{(0)}_{n,j} f - f \rangle| \\ &\leq \|\tilde{f}\| \|f - P^{(0)}_{n,j} f\| + c\|\tilde{f}\| \|f - P^{(0)}_{n,j} f\| \to 0, \end{aligned}$$

as $j \to \infty$. □

For example, let

$$\tilde{f}(x) := \begin{cases} x + 1; & x \in [-1, 0[, \\ -x + 1; & x \in [0, 1[, \\ 0; & \text{otherwise} \end{cases}$$

and

$$f(x) := \begin{cases} -2x-2; & x \in [-1,0[, \\ 2x-2; & x \in [0,1[, \\ 0; & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{R}$. Now

$$\langle \tilde{P}_{1,j}\tilde{f}, f\rangle \to \langle \tilde{f}, f\rangle = \int_{\mathbb{R}} \tilde{f}(x)f(x)dx = -\frac{4}{3}$$

as $j \to \infty$.

6.2. *Tensor product representation of the dual MRA*

It follows from Eq. (2.1) that $C_0(\mathbb{R}^n, K)^* \hat{\otimes}_{\varepsilon^s} C_0(\mathbb{R}, K)^*$ is a closed subspace of $C_0(\mathbb{R}^{n+1}, K)^*$.

Definition 6.7. Let $j \in \mathbb{Z}$. Define

$$\tilde{M}_{n,j} :=_{\text{n.s.}} \hat{\bigotimes_{k=1}^{n}}_{(C_0(\mathbb{R}^k,K)^*)} \tilde{V}_{1,j}.$$

Banach space $\tilde{M}_{n,j}$ is a closed subspace of $C_0(\mathbb{R}^n, K)^*$.

Definition 6.8. Define function

$$\xi^{[n]} : \hat{\bigotimes_{k=1}^{n}}_{\pi} l^1(\mathbb{Z}, K) \to l^1(\mathbb{Z}^n, K)$$

by

$$\xi^{[n]}(\check{\mathbf{e}}_{\mathbf{k}}^{\otimes}) := \check{\mathbf{e}}_{\mathbf{k}}, \quad \mathbf{k} \in \mathbb{Z}^n,$$

and extending by linearity and continuity onto whole $\hat{\bigotimes}_{k=1}^{n}{}_{\pi} l^1(\mathbb{Z}, K)$.

Function $\xi^{[n]}$ is an isometric isomorphism from Banach space $\hat{\bigotimes}_{k=1}^{n}{}_{\pi} l^1(\mathbb{Z}, K)$ onto Banach space $l^1(\mathbb{Z}^n, K)$.

Lemma 6.2. *When* $j \in \mathbb{Z}$

$$\hat{\bigotimes_{k=1}^{n}}_{\pi} \tilde{V}_{1,j} =_{\text{n.s.}} \tilde{V}_{n,j}.$$

Proof. Let

$$E :=_{\text{n.s.}} \hat{\bigotimes_{k=1}^{n}}_{\pi} \tilde{V}_{1,j}$$

and

$$F :=_{\text{n.s.}} \hat{\bigotimes_{k=1}^{n}}_{\pi} l^1(\mathbb{Z}, K).$$

Let

$$\alpha := \bigotimes_{k=1}^{n}{}_{\pi} \tilde{\iota}_{1,j}$$

Function α is an isometric isomorphism from F onto E. Let $\beta := \alpha \circ (\xi^{[n]})^{-1}$. Now β is an isometric isomorphism from $l^1(\mathbb{Z}^n, K)$ onto E and $\beta(\check{\mathbf{e}}_{\mathbf{k}}) = \tilde{\varphi}^{[n]}_{j,\mathbf{k}} = \tilde{\iota}_{n,j}(\check{\mathbf{e}}_{\mathbf{k}})$ for all $\mathbf{k} \in \mathbb{Z}^n$. When $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$ we have $\beta(\mathbf{d}) = \tilde{\iota}_{n,j}(\mathbf{d})$ and $\|\beta(\mathbf{d})\|_E = \|\tilde{\iota}_{n,j}(\mathbf{d})\|_{C_0(\mathbb{R}^n,K)^*}$. □

Lemma 6.3. *When* $j \in \mathbb{Z}$

$$\tilde{V}_{n,j} =_{\text{n.s.}} \tilde{M}_{n,j} =_{\text{n.s.}} \hat{\bigotimes_{k=1}^{n}}_{\pi} \tilde{V}_{1,j} =_{\text{n.s.}} \hat{\bigotimes_{k=1}^{n}}_{\varepsilon^s} \tilde{V}_{1,j}.$$

Proof. Use induction by n, metric approximation property of l^1, and Proposition 7.1 in Ref. 41. □

Definition 6.9. Define

$$\tilde{N}_{n,\mathbf{s},j} :=_{\text{n.s.}} \hat{\bigotimes_{k=1}^{n}}_{(C_0(\mathbb{R}^n,K)^*)} \tilde{W}_{1,\mathbf{s}[k],j},$$

where $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$.

Definition 6.10. Let $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$. Define operator $\tilde{R}_{n,\mathbf{s},j} : \tilde{V}_{n,j+1} \to \tilde{N}_{n,\mathbf{s},j}$ by

$$\tilde{R}_{n,\mathbf{s},j} = \bigotimes_{k=1}^{n}{}_{(C_0(\mathbb{R}^k,K)^*, C_0(\mathbb{R}^k,K)^*)} (\tilde{Q}_{1,\mathbf{s}[k],j}|\tilde{V}_{1,j+1}).$$

Assume that $j \in \mathbb{Z}$, and $\mathbf{s} \in \{0,1\}^n$. Now

$$\tilde{R}_{n,\mathbf{s},j}\tilde{f} = \sum_{\mathbf{k}\in\mathbb{Z}^n} \langle \tilde{f}, \psi^{[n]}_{\mathbf{s},j,\mathbf{k}} \rangle \tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}} \tag{6.1}$$

for all $\tilde{f} \in \tilde{V}_{n,j+1}$. The series in Eq. (6.1) converges absolutely for all $\tilde{f} \in \tilde{V}_{n,j+1}$. We also have $\tilde{W}_{n,\mathbf{s},j} =_{\text{n.s.}} \tilde{N}_{n,\mathbf{s},j}$ and $\tilde{Q}_{n,\mathbf{s},j} = \tilde{R}_{n,\mathbf{s},j} \circ \tilde{P}_{n,j+1}$.

7. Besov Space Norm Equivalence

7.1. *Norm equivalence for the Besov spaces in the n-dimensional case*

This derivation is based on the corresponding one-dimensional derivation in Ref. 17. The cases $p < 1$ or $q < 1$ yielding quasi-Banach spaces $B^{\sigma}_{p,q}$ are not discussed in this paper. We assume that $n \in \mathbb{Z}_+$ throughout this section.

We give first some definitions similar to those in Ref. 38 related to orthonormal wavelets.

Definition 7.1. Let $\bar{\varphi} : \mathbb{R}^n \to K$ be a mother scaling function of an orthonormal wavelet family. When $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$, and $f \in L^\infty(\mathbb{R}^n)$ define

$$\bar{\alpha}_{j,\mathbf{k}}(f) := 2^{nj} \int_{\mathbf{x}\in\mathbb{R}^n} \bar{\varphi}^*(2^j\mathbf{x} - \mathbf{k}) f(\mathbf{x}) d\tau.$$

The spaces $V_j(p)$ that are defined in Ref. 38 are denoted by $\bar{V}_j(p)$ in this document. Space $\bar{V}_j(p)$ is a closed subspace of $L^p(\mathbb{R}^n)$ for each $j \in \mathbb{Z}$ and $p \in [1, \infty]$ at least when the mother scaling function $\bar{\varphi}$ is continuous and compactly supported.

Definition 7.2. Let $\bar{\varphi} : \mathbb{R}^n \to \mathbb{C}$ be a scaling function of an orthonormal wavelet family. Let $p \in [1, \infty]$ and $j \in \mathbb{Z}$. Define operator $\bar{P}_j^{(p)} : L^p(\mathbb{R}^n) \to \bar{V}_j(p)$ by

$$(\bar{P}_j^{(p)} f)(\mathbf{x}) := \sum_{\mathbf{k}\in\mathbb{Z}^n} \bar{\alpha}_{j,\mathbf{k}}(f) \bar{\varphi}(2^j\mathbf{x} - \mathbf{k})$$

for all $f \in L^p(\mathbb{R}^n)$ and $\mathbf{x} \in \mathbb{R}^n$. Define also $\bar{Q}_j^{(p)} := \bar{P}_{j+1}^{(p)} - \bar{P}_j^{(p)}$.

When $\bar{\varphi}$ is a compactly supported and continuous function operator $\bar{P}_j^{(p)}$ is a continuous linear projection of $L^p(\mathbb{R}^n)$ onto $\bar{V}_j(p)$ for each $j \in \mathbb{Z}$ and $p \in [1, \infty]$.

Definition 7.3. Let $p \in [1, \infty]$, $q \in [1, \infty]$, and $\sigma \in \mathbb{R}_+$. Let $j_0 \in \mathbb{Z}$ and $\bar{r} \in \mathbb{Z}_+$, $\bar{r} > \sigma$. Let $\bar{\varphi} : \mathbb{R}^n \to \mathbb{C}$ be a mother scaling function of an $\bar{r}$-regular orthonormal MRA of $L^2(\mathbb{R}^n)$ and $\bar{P}_j^{(p)}$ and $\bar{Q}_j^{(p)}$, $j \in \mathbb{Z}$, be the corresponding projection operators. When $f \in L^p(\mathbb{R}^n)$ define

$$\|f\|^{(o)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0} := \|\bar{P}_{j_0}^{(p)} f | L^p(\mathbb{R}^n)\| + \left\|\bar{\mathbf{h}} | l^q(\mathbb{N} + j_0)\right\|,$$

where

$$\bar{h}_j := 2^{j\sigma} \|\bar{Q}_j^{(p)} f | L^p(\mathbb{R}^n)\|, \quad j \in \mathbb{N} + j_0$$
$$\bar{\mathbf{h}} := (\bar{h}_j)_{j=j_0}^\infty.$$

Norm $\|\cdot\|^{(o)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ is an equivalent norm for the Besov space $B^\sigma_{p,q}(\mathbb{R}^n)$ and characterizes $B^\sigma_{p,q}(\mathbb{R}^n)$ on $\mathrm{bor}(\mathbb{R}^n, \mathbb{C})$. This has been proved in Chap. 2.9, Proposition 4 in Ref. 38.

Definition 7.4. Let $p \in [1, \infty]$, $q \in [1, \infty]$, and $\sigma \in \mathbb{R}_+$. Let $j_0 \in \mathbb{Z}$. Let $P^{(\mathrm{u})}_{n,j}$ and $Q^{(\mathrm{u})}_{n,j}$, $j \in \mathbb{Z}$, be the projection operators belonging to a compactly supported interpolating tensor product MRA of $C_{\mathrm{u}}(\mathbb{R}^n)$. When $f \in C_{\mathrm{u}}(\mathbb{R}^n)$ define

$$\|f\|^{(i)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0} := \|P^{(\mathrm{u})}_{n,j_0} f | L^p(\mathbb{R}^n)\| + \|\mathbf{h}|l^q(\mathbb{N}+j_0)\|,$$

where

$$h_j := 2^{j\sigma}\|Q^{(\mathrm{u})}_{n,j} f | L^p(\mathbb{R}^n)\|, \quad j \in \mathbb{N}+j_0,$$

$$\mathbf{h} := (h_j)_{j=j_0}^{\infty}.$$

Definition 7.5. Let $p \in [1,\infty]$, $q \in [1,\infty]$, and $\sigma \in \mathbb{R}_+$. Let $j_0 \in \mathbb{Z}$. Let $\tilde{\varphi}^{[n]}_{j,\mathbf{k}}$ and $\tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}$, $j \in \mathbb{Z}$, $\mathbf{s} \in J_+(n)$, $\mathbf{k} \in \mathbb{Z}^n$, be the dual scaling functions and dual wavelets belonging to a compactly supported interpolating tensor product MRA of $C_{\mathrm{u}}(\mathbb{R}^n)$. When $f \in C_{\mathrm{u}}(\mathbb{R}^n)$ define

$$\|f\|^{(w)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0} := \|(\langle\tilde{\varphi}^{[n]}_{j_0,\mathbf{k}}, f\rangle)_{\mathbf{k}\in\mathbb{Z}^n}\|_p + \|(2^{(\sigma-\frac{n}{p})j}\|(\langle\tilde{\psi}^{[n]}_{\mathbf{s},j,\ell}, f\rangle)_{\mathbf{s}\in J_+(n),\ell\in\mathbb{Z}^n}\|_p)_{j=j_0}^{\infty}\|_q.$$

Definition 7.6. When $j \in \mathbb{Z}$ and $p \in [1,\infty]$ define

$$V^{(u)}_{n,j}(p) := \left\{\mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\varphi^{[n]}_{j,\mathbf{k}}(\mathbf{x}) : \mathbf{a} \in l^p(\mathbb{Z}^n)\right\}$$

and $\|f|V^{(u)}_{n,j}(p)\| := \|f|L^p(\mathbb{R}^n)\|$ for all $f \in V^{(u)}_{n,j}(p)$.

Definition 7.7. Let $j \in \mathbb{Z}$ and $p \in [1,\infty]$. When $T \in \mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ define

$$\|T\|_{V^{(u)}_{n,j}(p)} := \|(T|_{V^{(u)}_{n,j}(p)})|\mathcal{L}(V^{(u)}_{n,j}(p), L^p(\mathbb{R}^n))\|.$$

Definition 7.8. Let $j \in \mathbb{Z}$ and $p \in [1,\infty]$. When $T \in \mathcal{L}(C_{\mathrm{u}}(\mathbb{R}^n), C_{\mathrm{u}}(\mathbb{R}^n))$ and $T[\bar{V}_j(p)] \subset L^p(\mathbb{R}^n)$ define

$$\|T\|_{\bar{V}_j(p)} := \|(T|_{\bar{V}_j(p)})|\mathcal{L}(\bar{V}_j(p), L^p(\mathbb{R}^n))\|.$$

Definition 7.9. When $p \in [1,\infty]$ define

$$c_{\mathrm{interp}}(p) := \begin{cases} \dfrac{p}{p-1}; & p \in]1,\infty[, \\ 1; & p = 1 \vee p = \infty. \end{cases}$$

Lemma 7.1. *Suppose that $q \in [1,\infty]$ and $t \in \mathbb{R}_+$. Let $a(j,j') := 2^{-|j'-j|t}$ for all $j, j' \in \mathbb{N}$. Define*

$$A\mathbf{b} := \left(\sum_{j'=0}^{\infty} a(j,j')b_{j'}\right)_{j\in\mathbb{N}}$$

for all $\mathbf{b} \in l^q(\mathbb{N})$. Then $A \in \mathcal{L}(l^q(\mathbb{N}), l^q(\mathbb{N}))$.

Proof. Cases $q = 1$ and $q = \infty$ can be proved by starting from the expression of $A\mathbf{b}$ where $\mathbf{b} \in l^q(\mathbb{N})$. When $q \in]1, \infty[$ the proof is based on interpolation of Banach spaces, see Ref. 2. □

Theorem 7.1. *Let $p \in [1, \infty]$, $q \in [1, \infty]$, $j_0 \in \mathbb{Z}$, $\sigma \in \mathbb{R}_+$, and $n/p < \sigma < r_0$. Let $\varphi^{[n]}$ be a scaling function of a compactly supported interpolating tensor product MRA of $C_{\mathrm{u}}(\mathbb{R}^n)$ and $\bar{\varphi} : \mathbb{R}^n \to \mathbb{C}$ a compactly supported and continuous mother scaling function of a $(\lfloor\sigma\rfloor + 1)$-regular orthonormal wavelet family. Suppose that $\varphi^{[n]} \in C^{r_0}(\mathbb{R}^n)$. Suppose also that $(\tilde{\varphi}^{[n]}, \varphi^{[n]})$ and $(2^{nj}\bar{\varphi}^*, \bar{\varphi})$ span all polynomials of degree at most $\lceil\sigma\rceil - 1$. Let B be the normed space $B^\sigma_{p,q}(\mathbb{R}^n) \cap C_{\mathrm{u}}(\mathbb{R}^n)$ equipped with some Besov space norm. Then $\|\cdot\|^{(i)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ and $\|\cdot\|^{(o)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ are equivalent norms on the vector space B and norm $\|\cdot\|^{(i)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ characterizes B on $C_{\mathrm{u}}(\mathbb{R}^n)$.*

Proof. Let $\bar{r}_0 := \lfloor\sigma\rfloor + 1$. We prove first that $\|f\|^{(i)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0} \leq c\|f\|^{(o)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ for all $f \in B$ and for some $c \in \mathbb{R}_+$. Define q' by

$$\frac{1}{q'} + \frac{1}{q} = 1.$$

There exists $c_1 \in \mathbb{R}_+$ so that

$$\|P^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j'}(p)} \leq c_1 \cdot 2^{n(j'-j)/p} \tag{7.1}$$

for all $j, j' \in \mathbb{Z}$, $j' \geq j$. There exists $c_2 \in \mathbb{R}_+$ so that

$$\|I - P^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j'}(p)} \leq c_2 \cdot c_{\mathrm{interp}}(p) \cdot 2^{(j'-j)\bar{r}_0} \tag{7.2}$$

for all $j, j' \in \mathbb{Z}$, $j' \leq j$. These two results can be proved by Banach space interpolation between cases $p = 1$ and $p = \infty$. Let $f \in B$. We have

$$\begin{aligned}
\|P^{(\mathrm{u})}_{n,j_0} f\|_p &= \left\| P^{(\mathrm{u})}_{n,j_0} \left(\bar{P}^{(p)}_{j_0} f + \sum_{j=j_0}^{\infty} \bar{Q}^{(p)}_j f \right) \right\|_p \\
&\leq \|P^{(\mathrm{u})}_{n,j_0}\|_{\bar{V}_{j_0}(p)} \|\bar{P}^{(p)}_{j_0} f\|_p + \sum_{j=j_0}^{\infty} \|P^{(\mathrm{u})}_{n,j_0}\|_{\bar{V}_{j+1}(p)} \|\bar{Q}^{(p)}_j f\|_p \\
&\leq c_3 \|\bar{P}^{(p)}_{j_0} f\|_p + c_3 \cdot 2^{(n-nj_0)/p} \sum_{j=j_0}^{\infty} 2^{-j(\sigma - \frac{n}{p})} \bar{\mathbf{h}}[j],
\end{aligned}$$

where $\bar{\mathbf{h}} := (2^{j\sigma}\|\bar{Q}^{(p)}_j f\|_p)_{j=j_0}^{\infty}$. Let

$$c_4 := c_3 \cdot 2^{(n-nj_0)/p} \cdot \|(2^{-j'(\sigma - \frac{n}{p})})_{j'=j_0}^{\infty}\|_{q'}.$$

Now

$$\|P^{(\mathrm{u})}_{n,j_0} f\|_p \leq c_3 \|\bar{P}^{(p)}_{j_0} f\|_p + c_4 \|\bar{\mathbf{h}}\|_q \leq \max\{c_3, c_4\} \cdot \|f\|^{(o)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0},$$

where c_3 and c_4 do not depend on f. We have

$$\|Q^{(\mathrm{u})}_{n,j} f\|_p = \left\| Q^{(\mathrm{u})}_{n,j} \left(\bar{P}^{(p)}_{j_0} f + \sum_{j'=j_0}^{\infty} \bar{Q}^{(p)}_{j'} f \right) \right\|_p$$
$$\leq \|Q^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j_0}(p)} \|\bar{P}^{(p)}_{j_0} f\|_p + \sum_{j'=j_0}^{\infty} \|Q^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j'+1}(p)} \|\bar{Q}^{(p)}_{j'} f\|_p$$

for all $j \in \mathbb{Z}$, $j \geq j_0$. It follows from Eq. (7.2) that

$$2^{j\sigma} \|Q^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j_0}(p)} \|\bar{P}^{(p)}_{j_0} f\|_p \leq c_5 \cdot 2^{-j(\bar{r}_0 - \sigma)} \|\bar{P}^{(p)}_{j_0} f\|_p$$

for all $j \in \mathbb{Z}$, $j \geq j_0$, where c_5 is independent of f. Consequently

$$\|(2^{j\sigma} \|Q^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j_0}(p)} \|\bar{P}^{(p)}_{j_0} f\|_p)_{j=j_0}^{\infty}\|_q \leq c_5 \|(2^{-j(\bar{r}_0-\sigma)})_{j=j_0}^{\infty}\|_q \|\bar{P}^{(p)}_{j_0} f\|_p$$
$$= c_6 \|\bar{P}^{(p)}_{j_0} f\|_p \leq c_6 \|f\|^{(o)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0}. \tag{7.3}$$

When $j' < j$ it follows from Eq. (7.2) that

$$2^{j\sigma} \|Q^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j'+1}(p)} \|\bar{Q}^{(p)}_{j'} f\|_p. \leq c_7 \cdot 2^{-|j'-j|(\bar{r}_0-\sigma)} \bar{\mathbf{h}}[j']. \tag{7.4}$$

When $j' \geq j$ it follows from Eq. (7.1) that

$$2^{j\sigma} \|Q^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j'+1}(p)} \|\bar{Q}^{(p)}_{j'} f\|_p \leq c_8 \cdot 2^{-|j'-j|(\sigma - \frac{n}{p})} \bar{\mathbf{h}}[j']. \tag{7.5}$$

Let

$$w := \min\left\{\sigma - \frac{n}{p}, \bar{r}_0 - \sigma\right\}$$

and $c_9 := \max\{c_7, c_8\}$. Now by Eqs. (7.4) and (7.5)

$$a := \left\| \left(2^{j\sigma} \sum_{j'=j_0}^{\infty} \|Q^{(\mathrm{u})}_{n,j}\|_{\bar{V}_{j'+1}(p)} \|\bar{Q}^{(p)}_{j'} f\|_p \right)_{j=j_0}^{\infty} \right\|_q$$
$$\leq c_9 \left\| \left(\sum_{j'=j_0}^{\infty} 2^{-|j'-j|w} \bar{\mathbf{h}}[j'] \right)_{j=j_0}^{\infty} \right\|_q .$$

It follows from Lemma 7.1 that

$$a \leq c_{10} \|\bar{\mathbf{h}}\|_q \leq c_{10} \|f\|^{(o)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0}, \tag{7.6}$$

where c_{10} depends only on w and j_0. By Equations (7.3) and (7.6) we have $\|f\|^{(i)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0} \leq c_{10} \|f\|^{(o)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0}$.

We prove then that $\|g\|^{(o)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0} \leq c\|g\|^{(i)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0}$ for all $g \in B$ and for some $c \in \mathbb{R}_+$. Now

$$\begin{aligned}
\|P^{(\mathrm{u})}_{n,j_0}g\|_p + \sum_{j=j_0}^{\infty} \|Q^{(\mathrm{u})}_{n,j}g\|_p &= \|P^{(\mathrm{u})}_{n,j_0}g\|_p + \sum_{j=j_0}^{\infty} \|2^{-j\sigma} \cdot 2^{j\sigma} Q^{(\mathrm{u})}_{n,j}g\|_p \\
&\leq \|P^{(\mathrm{u})}_{n,j_0}g\|_p + \|(2^{j\sigma}\|Q^{(\mathrm{u})}_{n,j}g\|_p)_{j=j_0}^{\infty}\|_q \sum_{j=j_0}^{\infty} 2^{-j\sigma} \\
&\leq c_{11}\|g\|^{(i)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0} \leq c_{12}\|g\|^{(o)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0} \in \mathbb{R}_0,
\end{aligned}$$

where c_{11} and c_{12} do not depend on g. Consequently

$$P^{(\mathrm{u})}_{n,j_0}g + \sum_{j=j_0}^{m} Q^{(\mathrm{u})}_{n,j}g \to g$$

in $L^p(\mathbb{R}^n)$ as $m \to \infty$. We have

$$\begin{aligned}
\|\bar{P}^{(p)}_{j_0}g\|_p &= \left\|\bar{P}^{(p)}_{j_0}\left(P^{(\mathrm{u})}_{n,j_0}g + \sum_{j=j_0}^{\infty} Q^{(\mathrm{u})}_{n,j}g\right)\right\|_p \\
&\leq \|\bar{P}^{(p)}_{j_0}\|\|P^{(\mathrm{u})}_{n,j_0}g\|_p + \sum_{j'=j_0}^{\infty} \|\bar{P}^{(p)}_{j_0}\|\|Q^{(\mathrm{u})}_{n,j}g\|_p \\
&\leq c_{13}\|P^{(\mathrm{u})}_{n,j_0}g\|_p + c_{13}\sum_{j=j_0}^{\infty} 2^{-j\sigma}\mathbf{h}[j] \\
&\leq c_{13}\|P^{(\mathrm{u})}_{n,j_0}g\|_p + c_{13}\|(2^{-j\sigma})_{j=j_0}^{\infty}\|_{q'}\|\mathbf{h}\|_q \\
&\leq c_{14}\|g\|^{(i)}_{B^{\sigma}_{p,q}(\mathbb{R}^n);j_0},
\end{aligned}$$

where $\mathbf{h} := (2^{j\sigma}\|Q^{(\mathrm{u})}_{n,j}g\|_p)_{j=j_=}^{\infty}$. There exists $c_{15} \in \mathbb{R}_+$ so that

$$\|I - \bar{P}^{(p)}_j\|_{V^{(u)}_{n,j'}(p)} \leq c_{15} \cdot c_{\mathrm{interp}}(p) \cdot 2^{(j'-j)\sigma} \tag{7.7}$$

for all $j, j' \in \mathbb{Z}$, $j' \leq j$. This can be proved by Banach space interpolation between cases $p = 1$ and $p = \infty$. Furthermore,

$$\begin{aligned}
\|\bar{Q}^{(p)}_j g\|_p &= \left\|\bar{Q}^{(p)}_j\left(P^{(\mathrm{u})}_{n,j_0}g + \sum_{j'=j_0}^{\infty} Q^{(\mathrm{u})}_{n,j'}g\right)\right\|_p \\
&\leq \|\bar{Q}^{(p)}_j\|\|P^{(\mathrm{u})}_{n,j_0}g\|_p + \sum_{j'=j_0}^{\infty} \|\bar{Q}^{(p)}_j\|\|Q^{(\mathrm{u})}_{n,j'}g\|_p.
\end{aligned}$$

It follows from Eq. (7.7) that

$$2^{j\sigma}\|\bar{Q}_j^{(p)}\|\|P_{n,j_0}^{(\mathrm{u})}g\|_p \leq c_{16}\cdot 2^{-j(r_0-\sigma)}\|P_{n,j_0}^{(\mathrm{u})}g\|_p$$

for all $j \geq j_0$ where c_{16} is independent of g. Consequently

$$\begin{aligned}\|(2^{j\sigma}\|\bar{Q}_j^{(p)}\|\|P_{n,j_0}^{(\mathrm{u})}g\|_p)_{j=j_0}^{\infty}\|_q &\leq c_{16}\|(2^{-j(r_0-\sigma)})_{j=j_0}^{\infty}\|_q\|P_{n,j_0}^{(\mathrm{u})}g\|_p \\ &= c_{17}\|P_{n,j_0}^{(\mathrm{u})}g\|_p \leq c_{17}\|g\|_{B_{p,q}^{\sigma}(\mathbb{R}^n);j_0}^{(i)}. \end{aligned} \tag{7.8}$$

When $j' < j$ it follows from Eq. (7.7) that

$$2^{j\sigma}\|\bar{Q}_j^{(p)}\|\|Q_{n,j'}^{(\mathrm{u})}g\|_p \leq c_{18}\cdot 2^{-|j'-j|(r_0-\sigma)}\mathbf{h}[j']. \tag{7.9}$$

When $j' \geq j$ it follows from uniform boundedness of the projection operators that

$$2^{j\sigma}\|\bar{Q}_j^{(p)}\|\|Q_{n,j'}^{(\mathrm{u})}g\|_p \leq c_{19}\cdot 2^{-|j'-j|\sigma}\mathbf{h}[j']. \tag{7.10}$$

Let

$$z := \min\{\sigma, r_0-\sigma\}.$$

Now by Eqs. (7.9) and (7.10) we have

$$\begin{aligned} b &:= \left\|\left(\sum_{j'=j_0}^{\infty} 2^{j\sigma}\|\bar{Q}_j^{(p)}\|\|Q_{n,j'}^{(\mathrm{u})}g\|_p\right)_{j=j_0}^{\infty}\right\|_q \\ &\leq c_{20}\left\|\left(\sum_{j'=j_0}^{\infty} 2^{-|j'-j|z}\mathbf{h}[j']\right)_{j=j_0}^{\infty}\right\|_q. \end{aligned}$$

It follows from Lemma 7.1 that

$$b \leq c_{21}\|\mathbf{h}\|_q \leq c_{21}\|f\|_{B_{p,q}^{\sigma}(\mathbb{R}^n);j_0}^{(i)}, \tag{7.11}$$

where c_{21} depends only on z and j_0. By Eqs. (7.8) and (7.11) we have $\|g\|_{B_{p,q}^{\sigma}(\mathbb{R}^n);j_0}^{(o)} \leq c_{22}\|g\|_{B_{p,q}^{\sigma}(\mathbb{R}^n);j_0}^{(i)}$.

Suppose then that $\|g\|_{B_{p,q}^{\sigma}(\mathbb{R}^n);j_0}^{(o)} = \infty$. Let

$$A\mathbf{b} := \left(\sum_{j'=j_0}^{\infty} 2^{-|j'-j|z}\mathbf{b}[j']\right)_{j=j_0}^{\infty}, \quad \mathbf{b} \in \mathbb{C}^{\mathbb{N}+j_0}.$$

Now

$$\infty = \|(2^{j\sigma}\|\bar{Q}_j^{(p)}g\|_p)_{j=j_0}^{\infty}\|_q \leq c_{17}\|P_{n,j_0}^{(\mathrm{u})}g\|_p + c_{20}\|A\mathbf{h}\|_q.$$

By the continuity of $A|_{l^q(\mathbb{N}+j_0)}$ we must have $\|\mathbf{h}\|_q = \infty$. Consequently $\|g\|_{B_{p,q}^{\sigma}(\mathbb{R}^n);j_0}^{(i)} = \infty$. □

It is possible to construct an orthonormal compactly supported Daubechies wavelet family with arbitrary high regularity $\bar{r} \in \mathbb{R}_+$ and polynomial span (number of vanishing moments) $\bar{d} \in \mathbb{N}$.[8] Consequently, when $j_0 \in \mathbb{Z}$, $\sigma \in \mathbb{R}_+$, $p \in [1,\infty]$, $q \in [1,\infty]$, $n/p < \sigma < r_0$, $\varphi^{[n]} \in C^{r_0}(\mathbb{R}^n)$, and $(\tilde{\varphi}^{[n]}, \varphi^{[n]})$ spans all polynomials of degree at most $\lceil \sigma \rceil - 1$ function $\varphi^{[n]}$ defines an equivalent norm $\| \cdot \|^{(i)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ on Banach space B.

Theorem 7.2. *Let $p \in [1,\infty]$, $q \in [1,\infty]$, $\sigma \in \mathbb{R}_+$, $j_0 \in \mathbb{Z}$, $r_0 \in \mathbb{R}_+$, and $n/p < \sigma < r_0$. Let $\varphi^{[n]}$ be a scaling function of a compactly supported interpolating tensor product MRA of $C_{\mathrm{u}}(\mathbb{R}^n)$ and suppose that $\varphi^{[n]} \in C^{r_0}(\mathbb{R}^n)$. Let B be the normed space $B^\sigma_{p,q}(\mathbb{R}^n) \cap C_{\mathrm{u}}(\mathbb{R}^n)$ equipped with some Besov space norm. Then $\| \cdot \|^{(i)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ and $\| \cdot \|^{(w)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ are equivalent to the norm of B and they characterize B on $C_{\mathrm{u}}(\mathbb{R}^n)$.*

Proof. Define

$$\mathbf{a}_j(f) := (\langle \tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}, f \rangle)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n},$$

where $f \in B$ and $j \in \mathbb{Z}$. By applying Banach space interpolation to cases $p = 1$ and $p = \infty$ we get

$$\begin{aligned} c_1 \cdot 2^{-\frac{nj_0}{p}} \left\| (\langle \tilde{\varphi}^{[n]}_{j_0,\mathbf{k}}, f \rangle)_{\mathbf{k} \in \mathbb{Z}^n} \right\|_p &\leq \| P^{(\mathrm{u})}_{n,j_0} f \| \\ &\leq c_2 \cdot 2^{-\frac{nj_0}{p}} \left\| (\langle \tilde{\varphi}^{[n]}_{j_0,\mathbf{k}}, f \rangle)_{\mathbf{k} \in \mathbb{Z}^n} \right\|_p \end{aligned} \tag{7.12}$$

and

$$c_3 \cdot 2^{-\frac{nj}{p}} \| \mathbf{a}_j(f) \|_p \leq \left\| Q^{(\mathrm{u})}_{n,j} f \right\|_p \leq c_4 \cdot 2^{-\frac{nj}{p}} \| \mathbf{a}_j(f) \|_p$$

for all $j \in \mathbb{N} + j_0$ and $f \in B$. Consequently

$$\begin{aligned} c_3 \left\| (2^{j(\sigma - \frac{n}{p})} \| \mathbf{a}_j(f) \|_p)_{j=j_0}^\infty \right\|_q &\leq \left\| (2^{j\sigma} \| Q^{(\mathrm{u})}_{n,j} f \|_p)_{j=j_0}^\infty \right\|_q \\ &\leq c_4 \left\| (2^{j(\sigma - \frac{n}{p})} \| \mathbf{a}_j(f) \|_p)_{j=j_0}^\infty \right\|_q \end{aligned} \tag{7.13}$$

for all $f \in B$.

Let $c_5 := \min\{c_1 \cdot 2^{-nj_0/p}, c_3\}$ and $c_6 := \max\{c_2 \cdot 2^{-nj_0/p}, c_4\}$. It follows from Eqs. (7.12) and (7.13) that

$$c_5 \| f \|^{(w)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0} \leq \| f \|^{(i)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0} \leq c_6 \| f \|^{(w)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$$

for all $f \in B$. □

It is possible to construct Deslauriers–Dubuc wavelet families with arbitrary high regularity $r \in \mathbb{R}_+$ (i.e. $\varphi^{[n]} \in C^r(\mathbb{R}^n)$) and polynomial span $d \in \mathbb{N}$.[17] As a direct consequence of Theorems 7.1 and 7.2 norms $\| \cdot \|^{(w)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ and $\| \cdot \|^{(i)}_{B^\sigma_{p,q}(\mathbb{R}^n);j_0}$ are equivalent to the restriction of some norm of Besov space $B^\sigma_{p,q}(\mathbb{R}^n)$ onto the vector space $B^\sigma_{p,q}(\mathbb{R}^n) \cap C_{\mathrm{u}}(\mathbb{R}^n)$ and these norms characterize $B^\sigma_{p,q}(\mathbb{R}^n) \cap C_{\mathrm{u}}(\mathbb{R}^n)$ on $C_{\mathrm{u}}(\mathbb{R}^n)$.

7.2. *Consequences of the Besov space norm equivalence*

In particular, the Besov space norm equivalence holds for Hölder spaces $C^\sigma(\mathbb{R}^n)$ for $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. When $n \in \mathbb{Z}_+$, $\sigma \in \mathbb{R}_+$, and $q \in [1, \infty]$ we have $B^\sigma_{\infty,q}(\mathbb{R}^n) \subset C_{\mathrm{u}}(\mathbb{R}^n)$ (see Proposition 2.3.2/2(i) and Eq. (2.3.5/1) in Ref. 44).

Definition 7.10. When $n \in \mathbb{Z}_+$ and $j_0 \in \mathbb{Z}$ define

$$\Omega(n, j_0) := \{(\mathbf{0}_n, j_0, \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\} \cup \bigcup_{j=j_0}^{\infty} \bigcup_{\mathbf{s} \in J_+(n)} \{(\mathbf{s}, j, \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\}.$$

As the Banach space $B^\sigma_{\infty,\infty}(\mathbb{R}^n) =_{\mathrm{tvs}} \mathcal{Z}^\sigma(\mathbb{R}^n)$ is isomorphic to l^∞ (see Sec. 2.5.5, p. 87 in Ref. 44) the set $\{\psi_\alpha : \alpha \in \Omega(n, j_0)\}$ cannot be a Schauder basis of $B^\sigma_{\infty,\infty}(\mathbb{R}^n)$ with any summing order.

Remark 7.1. Suppose that $n \in \mathbb{Z}_+$, $j_0 \in \mathbb{Z}$, $\sigma \in \mathbb{R}_+$, and $\varphi^{[n]} \in C^r(\mathbb{R}^n)$ where $r \in \mathbb{R}_+$, $r > \sigma$. There exists $r_1 \in \mathbb{Z}_+$ so that $\operatorname{supp} \varphi^{[n]} \in \overline{B}_{\mathbb{R}^n}(0; r_1)$. Consider

$$f := \sum_{k=0}^{\infty} 2^{-(j_0+k)\sigma} \psi^{[n]}_{\mathbf{e}^{[n]}_1, j_0+k, \eta(k)},$$

where

$$c := 2r_1 + \lceil 2^{j_0} \rceil (2\lceil \sigma \rceil + 1),$$
$$\eta(k) := c \cdot 2^k k \mathbf{e}^{[n]}_1$$

for all $k \in \mathbb{Z}$, $k \geq j_0$, and the series converges in $C_0(\mathbb{R}^n)$. Let $m := \lceil r \rceil$ and

$$g_k := 2^{-(j_0+k)\sigma} \psi^{[n]}_{\mathbf{e}^{[n]}_1, j_0+k, \eta(k)}$$

for all $k \in \mathbb{N}$. As $\psi^{[n]}_{\mathbf{e}^{[n]}_1} \in B^\sigma_{\infty,\infty}(\mathbb{R}^n)$ it follows that there exists $c_1 \in \mathbb{R}_+$ so that

$$\omega^m_\infty(g_k; t) \leq c_1 t^\sigma \tag{7.14}$$

for all $k \in \mathbb{N}$ and $t \in]0, 1[$. Define $S_{j,\ell}(y) := \overline{B}_{\mathbb{R}^n}(2^{-j}\ell; 2^{-j} r_1 + y)$, where $j \in \mathbb{Z}$, $\ell \in \mathbb{Z}^n$, and $y \in \mathbb{R}_0$. Suppose that $t_1 \in]0, 1[$. We have

$$\operatorname{supp}_{\mathrm{set}} \Delta^m_{\mathbf{h}} g_k \subset S_{j_0+k,\eta(k)}(m) \tag{7.15}$$

for all $\mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; t_1)$ and $k \in \mathbb{N}$. Furthermore,

$$S_{j_0+k,\eta(k)}(m) \cap S_{j_0+l,\eta(l)}(m) = \emptyset \tag{7.16}$$

for all $k, l \in \mathbb{N}$ and $k \neq l$. Using Eqs. (7.15) and (7.16) we obtain

$$\omega^m_\infty(f; t_1) \leq \sup\{\omega^m_\infty(g_k; t_1) : k \in \mathbb{N}\}. \tag{7.17}$$

We have $f \in L^\infty(\mathbb{R}^n)$ and by Eqs. (7.14) and (7.17) $\omega^m_\infty(f; t_1) \leq c_2 t_1^\sigma$. Thus $f \in B^\sigma_{\infty,\infty}(\mathbb{R}^n)$.

Suppose that

$$f = \sum_{k=0}^{\infty} \mathbf{d}[\iota(k)]\psi_{\iota(k)}$$

for some sequence $\mathbf{d} \in \mathbb{C}^{\Omega(n,j_0)}$ where the series converges in $B^{\sigma}_{\infty,\infty}(\mathbb{R}^n)$. Define

$$\xi_b := \sum_{k=0}^{b} \mathbf{d}[\iota(k)]\psi_{\iota(k)}$$

where $b \in \mathbb{N}$. Let $b_1 \in \mathbb{N}$. Define

$$j_1 := \max\{j(\iota(k)) \in \mathbb{Z} : k \in Z_0(b_1)\} + 1.$$

There exists $k_1 \in \mathbb{N}$ so that $\iota(k_1) = \alpha(j_1 - j_0)$. Now $j_1 \neq j(\iota(k))$ for all $k \in Z_0(b_1)$ and consequently $\iota(k_1) \neq \iota(k)$ for all $k \in Z_0(b_1)$. It follows that $k_1 > b_1$. Let $k_2 := j_1 - j_0$. Now

$$\iota(k_1) = (\mathbf{e}_1^{[n]}, j_1, c \cdot 2^{k_2} k_2 \mathbf{e}_1^{[n]}).$$

We have

$$|\langle \tilde{\psi}_{\iota(k_1)}, f - \xi_{b_1} \rangle| \leq \|(\langle \tilde{\psi}^{[n]}_{\mathbf{s},j_1,\mathbf{k}}, f - \xi_{b_1} \rangle)_{\mathbf{s}\in J_+(n),\mathbf{k}\in\mathbb{Z}^n}\|_{\infty}$$

and $|\langle \tilde{\psi}^{[n]}_{\iota(k_1)}, f - \xi_{b_1} \rangle| = 2^{-(j_0+k_2)\sigma}$. Hence

$$\begin{aligned} 1 &\leq 2^{(j_0+k_2)\sigma} \|(\langle \tilde{\psi}^{[n]}_{\mathbf{s},j_1,\mathbf{k}}, f - \xi_{b_1} \rangle)_{\mathbf{s}\in J_+(n),\mathbf{k}\in\mathbb{Z}^n}\|_{\infty} \\ &\leq \|(2^{j\sigma} \|(\langle \tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}, f - \xi_{b_1} \rangle)_{\mathbf{s}\in J_+(n),\mathbf{k}\in\mathbb{Z}^n}\|_{\infty})_{j=j_0}^{\infty}\|_{\infty} \\ &\leq \|f - \xi_{b_1}\|^{(w)}_{B^{\sigma}_{\infty,\infty}(\mathbb{R}^n);j_0}. \end{aligned}$$

Thus $\xi_b \not\to f$ in Banach space $B^{\sigma}_{\infty,\infty}(\mathbb{R}^n)$ as $b \to \infty$.

Theorem 7.3. *Let $n \in \mathbb{Z}_+$, $\sigma \in \mathbb{R}_+$, $q \in [1, \infty[$, and $j_0 \in \mathbb{Z}$. Let $\varphi^{[n]}$ be a mother scaling function of a compactly supported tensor product MRA of $C_0(\mathbb{R}^n)$. Suppose that $\varphi^{[n]} \in C^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}_+$, $r > \sigma$. Then $\{\psi_\alpha : \alpha \in \Omega(n, j_0)\}$ is an unconditional basis of Banach space $B^{\sigma}_{\infty,q}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ equipped with a norm of $B^{\sigma}_{\infty,q}(\mathbb{R}^n)$ and the coefficient functional corresponding to basis vector ψ_α is $\tilde{\psi}_\alpha$ for each $\alpha \in \Omega(n, j_0)$.*

Proof. Let

$$\begin{aligned} S_j(m) &:= \{(\mathbf{0}_n, j, \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n, \|\mathbf{k}\|_2 \leq m\}, \quad j \in \mathbb{Z},\ m \in \mathbb{N}, \\ D_{\mathbf{s},j}(m) &:= \{(\mathbf{s}, j, \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n, \|\mathbf{k}\|_2 \leq m\}, \quad \mathbf{s} \in \{0,1\}^n,\ j \in \mathbb{Z},\ m \in \mathbb{N}, \\ D_j(m) &:= \bigcup_{\mathbf{s}\in J_+(n)} D_{\mathbf{s},j}(m), \quad j \in \mathbb{Z},\ m \in \mathbb{N}. \end{aligned}$$

Let

$$m_1 := \min\{m \in \mathbb{N} : \forall \mathbf{k} \in \mathbb{Z}^n : (\|\mathbf{k}\|_\infty > m \Rightarrow \forall \mathbf{s} \in \{0,1\}^n : \tilde{g}^{[n]}_{\mathbf{s},\mathbf{k}} = 0)\},$$
$$m_2 := \lceil m_1 \sqrt{n} \rceil,$$
$$A(k) := S_{j_0}(2^{j_0}k) \cup \bigcup_{l=j_0}^{j_0+k-1} D_l(m_2 + 2^{j_0+k}k), \quad k \in \mathbb{Z}_+.$$

Define $\mathbf{s}(\mathbf{a}) = \mathbf{s}'$, $j(\mathbf{a}) = j'$, and $\mathbf{k}(\mathbf{a}) = \mathbf{k}'$ for all $\mathbf{a} = (\mathbf{s}', j', \mathbf{k}')$, $\mathbf{s}' \in \{0,1\}^n$, $j' \in \mathbb{Z}$, and $\mathbf{k}' \in \mathbb{Z}^n$. Let $f \in B^\sigma_{\infty,q}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$. Let $\eta : \mathbb{N} \twoheadrightarrow \Omega(n, j_0)$ be a bijection. Let $\beta_\alpha := \langle \tilde{\psi}_\alpha, f \rangle$ for all $\alpha \in \Omega(n, j_0)$. Define

$$\beta^{(m)}_\alpha := \begin{cases} \langle \tilde{\psi}_\alpha, f \rangle; & \alpha \notin \eta[Z_0(m)], \\ 0; & \text{otherwise} \end{cases}$$

for all $m \in \mathbb{N}$ and $\alpha \in \Omega(n, j_0)$. Define

$$\xi_m := \sum_{\alpha \in \Omega(n,j_0)} \beta^{(m)}_\alpha \psi_\alpha$$

for all $m \in \mathbb{N}$. There exists $c_1 \in \mathbb{R}_+$ so that

$$|\langle \tilde{\psi}_\alpha, f \rangle| \le c_1 \|f\|_\infty \tag{7.18}$$

for all $f \in C_0(\mathbb{R}^n)$ and $\alpha \in \Omega(n, j_0)$.

Let $h \in \mathbb{R}_+$. Choose $j_1 \in \mathbb{Z}$, $j_1 > \max\{j_0, 0\}$ so that

$$\|(2^{j\sigma} \|(\langle \tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}, f \rangle)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n}\|_\infty)_{j=j_1}^\infty\|_q < \frac{h}{4}. \tag{7.19}$$

Choose $r_1 \in \mathbb{R}_+$ so that

$$\sup\{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| \ge r_1\} < \frac{h}{2^{n+2+j_1\sigma}(j_1 - j_0)c_1} \tag{7.20}$$

and let $m_3 := \max\{j_1 - j_0, \lceil r_1 \rceil\}$.

Choose $m_4 \in \mathbb{Z}_+$ so that $A(m_3) \subset \eta[Z_0(m_4)]$. Let $l \in \mathbb{N}$, $l > m_4$. If $\mathbf{s}(\eta(l)) = \mathbf{0}_n$ we have

$$\|(\beta^{(m_4)}_{\mathbf{0}_n,j_0,\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^n}\|_\infty < \frac{h}{4}. \tag{7.21}$$

If $\mathbf{s}(\eta(l)) \ne \mathbf{0}_n$ we have

$$\|(2^{j\sigma} \|(\beta^{(m_4)}_{\mathbf{s},j,\mathbf{k}})_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n}\|_\infty)_{j=j_0}^{j_1-1} | l^q(j_0 + \mathbb{N})\| < \frac{1}{4}h. \tag{7.22}$$

Thus by Eqs. (7.21) and (7.22), we get

$$\|(2^{j\sigma} \|(\beta^{(m_4)}_{\mathbf{s},j,\mathbf{k}})_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n}\|_\infty)_{j=j_0}^{\infty} | l^q(j_0 + \mathbb{N})\| < \frac{h}{2}. \tag{7.23}$$

By Eqs. (7.21) and (7.23) we have $\|\xi_{m_4}\|^{(w)}_{B^\sigma_{\infty,q}(\mathbb{R}^n);j_0} < h$. It follows that

$$s_m := \sum_{l=0}^m \beta_{\eta(l)} \psi_{\eta(l)} \to g$$

as $m \to \infty$ where the series converges in Banach space $B^\sigma_{\infty,q}(\mathbb{R}^n)$. We also have $s_m \to g$ as $m \to \infty$ in Banach space $C_0(\mathbb{R}^n)$. If we had $g \neq f$ we would have $\langle\tilde{\psi}_\gamma, f - g\rangle = \langle\tilde{\psi}_\gamma, f\rangle - \langle\tilde{\psi}_\gamma, g\rangle \neq 0$ for some $\gamma \in \Omega(n, j_0)$. Now $\langle\tilde{\psi}_\gamma, g\rangle = \beta_\gamma = \langle\tilde{\psi}_\gamma, f\rangle$, which is a contradiction. Hence $f = g$. □

Theorem 7.4. *Let $n \in \mathbb{Z}_+$ and $j_0 \in \mathbb{Z}$. Let $\sigma \in \mathbb{R}_+$, $p \in [1,\infty]$, and $q \in [1,\infty]$. Let $\varphi^{[n]}$ be a mother scaling function of a compactly supported tensor product MRA of $C_{\mathrm{u}}(\mathbb{R}^n)$. Suppose that $\varphi^{[n]} \in C^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}_+$, $r > \sigma$. Let $f \in B^\sigma_{p,q}(\mathbb{R}^n)$. Then*

$$f(\mathbf{x}) = \sum_{\alpha\in\Omega(n,j_0)} \langle\tilde{\psi}_\alpha, f\rangle\psi_\alpha(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and the series above converges absolutely for each $\mathbf{x} \in \mathbb{R}^n$.

Proof. Let

$$a := \left\|(2^{(\sigma-\frac{n}{p})j}\|(\langle\tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}, f\rangle)_{\mathbf{s}\in J_+(n),\mathbf{k}\in\mathbb{Z}^n}\|_p)_{j=j_0}^\infty\right\|_q.$$

Now

$$|\langle\tilde{\psi}^{[n]}_{\mathbf{s},j,\mathbf{k}}, f\rangle| \leq 2^{-(\sigma-\frac{n}{p})j}a \tag{7.24}$$

for all $\mathbf{s} \in J_+(n)$, $j \in \mathbb{N} + j_0$, and $\mathbf{k} \in \mathbb{Z}^n$. Let

$$A_0(\mathbf{x}) := \{(\mathbf{0}_n, j_0, \mathbf{k}) : \mathbf{x} \in \operatorname{supp}\varphi^{[n]}_{j_0,\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^n\}$$

for each $\mathbf{x} \in \mathbb{R}^n$ and

$$A_k(\mathbf{x}) := \{(\mathbf{s}, j_0 + k - 1, \mathbf{k}) : \mathbf{x} \in \operatorname{supp}\psi^{[n]}_{\mathbf{s},j_0+k-1,\mathbf{k}}, \mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n\}$$

for each $\mathbf{x} \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$. Let $\eta : \mathbb{N} \twoheadrightarrow \Omega(n, j_0)$ be a bijection and define

$$\beta^{(m)}_\alpha := \begin{cases} \langle\tilde{\psi}_\alpha, f\rangle; & \alpha \notin \eta[Z_0(m)], \\ 0; & \alpha \in \eta[Z_0(m)] \end{cases} \tag{7.25}$$

for each $m \in \mathbb{N}$ and $\alpha \in \Omega(n, j_0)$. Let

$$g_m(\mathbf{x}) := \sum_{k=0}^{m}\langle\tilde{\psi}_{\eta(k)}, f\rangle\psi_{\eta(k)}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $m \in \mathbb{N}$. It follows from Theorem 4.3 that

$$f(\mathbf{x}) = \sum_{k=0}^{\infty}\sum_{\alpha\in A_k(\mathbf{x})} \langle\tilde{\psi}_\alpha, f\rangle\psi_\alpha(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$. Define $c_1 := \max\{\|\psi^{[n]}_{\mathbf{s}}\|_\infty : \mathbf{s} \in \{0,1\}^n\}$. Let $\mathbf{y} \in \mathbb{R}^n$ and $h \in \mathbb{R}_+$. Define $m_1 := \max\{\#A_k(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$. Let

$$j_1 := \max\left\{j_0 + 1, \left\lceil\left(\sigma - \frac{n}{p}\right)^{-1}\log_2\frac{c_1 m_1 a}{(1 - 2^{-(\sigma-\frac{n}{p})h})}\right\rceil\right\}.$$

Choose $m_2 \in \mathbb{N}$ so that $A_k(\mathbf{y}) \subset \eta[Z_0(m_2)]$ for all $k \in Z_0(j_1 - j_0)$. Suppose that $m \in \mathbb{N}$, $m > m_2$, and $\alpha_0 \in A_{j_0+k}$ for some $k_0 \in Z_0(j_1 - j_0)$. By Eqs. (7.24) and

(7.25) we have

$$|f(\mathbf{y}) - g_m(\mathbf{y})| \leq c_1 \sum_{k=0}^{\infty} \sum_{\alpha \in A_k(\mathbf{y})} |\beta_\alpha^{(m)}| = c_1 \sum_{k=j_1-j_0+1}^{\infty} \sum_{\alpha \in A_k(\mathbf{y})} |\beta_\alpha^{(m)}|$$

$$\leq c_1 m_1 a \cdot 2^{-j_1\left(\sigma - \frac{n}{p}\right)} \frac{1}{1 - 2^{-\left(\sigma - \frac{n}{p}\right)}} < h.$$

Thus $g_m(\mathbf{x}) \to f(\mathbf{x})$ as $m \to \infty$ for all $\mathbf{x} \in \mathbb{R}^n$. □

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